# REAL AND INTEGRAL STRUCTURES IN QUANTUM COHOMOLOGY I: TORIC ORBIFOLDS

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Abstract. We study real and integral structures in the space of solutions to the quantum differential equations. First we show that, under mild conditions, any real structure in orbifold quantum cohomology yields a pure and polarized $tt^*$-geometry near the large radius limit. Secondly, we use mirror symmetry to calculate the "most natural" integral structure in quantum cohomology of toric orbifolds. We show that the integral structure pulled back from the singularity B-model is described only in terms of topological data in the A-model; $K$-group and a characteristic class. Using integral structures, we give a natural explanation why the quantum parameter should specialize to a root of unity in Ruan's crepant resolution conjecture.

Contents

## 1. Introduction

Quantum cohomology is a family of commutative algebra structures $(H^*(X,\mathbb{C}),\circ_\tau)$ on the cohomology parametrized by $\tau \in H^*(X,\mathbb{C})$. The structure constants of the quantum product $\circ_\tau$ are given by power series[1] in $\tau$ whose coefficients are genus zero Gromov-Witten invariants. The real or integral structures on quantum cohomology in the usual sense — the subspaces $H^*(X,\mathbb{R})$ or $H^*(X,\mathbb{Z})$ of $H^*(X,\mathbb{C})$ — are not the subject of the present paper. We will study hidden real or integral structures which lie in the space of solutions to *quantum differential equations*.

Our study of real or integral structures in quantum cohomology is motivated by mirror symmetry. Classical mirror symmetry for Calabi-Yau manifolds states that the Gromov-Witten theory (A-model) of a Calabi-Yau manifold $X$ is equivalent to the Hodge theory (B-model) of the mirror dual Calabi-Yau $X^\vee$. Small quantum cohomology of $X$ defines the *A-model variation of Hodge structures* (henceforth A-model VHS) on $\bigoplus_p H^{p,p}(X)$ [55, 27]. On the other hand, the deformation of complex structures of $X^\vee$ also defines a variation of Hodge structures (B-model VHS) on $H^n(X^\vee)$. Mathematically, mirror symmetry can be formulated as an isomorphism between the A-model VHS of $X$ and the B-model VHS of $X^\vee$. While the B-model VHS is naturally equipped with the integral local system $H^n(X^\vee,\mathbb{Z})$, the A-model VHS seems to lack such integral structures. Then we are led to the question: what is the natural integral structure on the A-model VHS? Our calculation for the toric orbifolds suggests that the $K$-group of $X$ should give the integral local system in the A-model VHS.

**Quantum cohomology as a VHS.** In this paper, we use the language of *semi-infinite variation of Hodge structures* due to Barannikov [7, 8] to include non Calabi-Yau case in our theory. We will briefly explain how this arises from quantum cohomology. It is well-known that quantum cohomology associates a one parameter family of flat connections, called Dubrovin connection, on the trivial vector bundle $H^*(X)\times H^*(X)\to H^*(X)$:

$$\nabla_i = \frac{\partial}{\partial t^i} + \frac{1}{z}\phi_i \circ_\tau .$$

Here, $\{\phi_i\}$ is a basis of $H^*(X)$, $\{t^i\}$ is a linear co-ordinate system on $H^*(X)$ dual to $\{\phi_i\}$ and $z\in\mathbb{C}^*$ is a parameter. Let $L(\tau,z)$ be the fundamental solution to the quantum differential equation $\nabla_i L(\tau,z)=0$ given by the gravitational descendants (see (27)). Here, $L(\tau,z)$ is an $\mathrm{End}(H^*(X))$-valued function in $(\tau,z)\in H^*(X)\times\mathbb{C}^*$. Following Coates-Givental [24], we introduce an infinite dimensional vector space $\mathcal{H}^X$ by

$$\mathcal{H}^X := H^*(X)\otimes\mathbb{C}\{z,z^{-1}\},$$

where $\mathbb{C}\{z,z^{-1}\}$ denotes the space of holomorphic functions on $\mathbb{C}^*$. Via the correspondence $\mathcal{H}^X \ni v(z)\mapsto L(\tau,z)v(z)$, we can think of $\mathcal{H}^X$ as the space of flat sections

[1]More precisely, Fourier series in the $H^2$-part of $\tau$ and power series in the other part of $\tau$.

of the Dubrovin connection. The fundamental solution $L(\tau,z)$ defines the family of "semi-infinite" subspaces of $\mathcal{H}^X$:

$$\mathbb{F}_\tau := L(\tau,z)^{-1}(H^*(X)\otimes\mathbb{C}\{z\}) \subset \mathcal{H}^X, \quad \tau\in H^*(X),$$

where $\mathbb{C}\{z\}$ denotes the space of holomorphic functions on $\mathbb{C}$. The semi-infinite flag $\cdots\subset z^{-1}\mathbb{F}_\tau\subset\mathbb{F}_\tau\subset z\mathbb{F}_\tau\subset\cdots$ satisfies properties analogous to the usual finite dimensional VHS:

$$\frac{\partial}{\partial t^i}\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau \qquad \text{(Griffiths Transversality)} \tag{1}$$

$$(\mathbb{F}_\tau,\mathbb{F}_\tau)_{\mathcal{H}^X}\subset\mathbb{C}\{z\} \quad \text{(Bilinear Relations)} \tag{2}$$

where $(\alpha,\beta)_{\mathcal{H}^X} = \int_X \alpha(-z)\cup\beta(z)$ for $\alpha,\beta\in\mathcal{H}^X$. We call this family of subspaces the *semi-infinite variation of Hodge structures* or $\frac{\infty}{2}$VHS.

**Real and integral structures.** The free $\mathbb{C}\{z,z^{-1}\}$-module $\mathcal{H}^X$ can be regarded as the space of global sections of the trivial vector bundle $\mathsf{H}^X$ over the space of parameters $z$:

$$\mathcal{H}^X = \Gamma(\mathbb{C}^*,\mathsf{H}^X), \quad \mathsf{H}^X = H^*(X)\times\mathbb{C}^*\to\mathbb{C}^*.$$

Due to the existence of the grading in quantum cohomology, Dubrovin connection $\nabla$ can be extended in the direction of the parameter $z$. The extended flat connection induces, via the fundamental solution $L(\tau,z)$, the following flat connection $\widehat{\nabla}_{z\partial_z}$ on $\mathsf{H}^X$:

$$\widehat{\nabla}_{z\partial_z} = z\partial_z + \mu - \frac{\rho}{z}\in\operatorname{End}_{\mathbb{C}}(\mathcal{H}^X), \quad \rho = c_1(X), \quad \mu \text{ is given in (23)}.$$

A *real or integral structure* (Definition 2.2) on the $\frac{\infty}{2}$VHS is given by the choice of a real or integral local system underlying the flat bundle $(\mathsf{H}^X,\widehat{\nabla}_{z\partial_z})$. A real structure defines a real subbundle of $\mathsf{H}^X|_{S^1}$ and an involution $\kappa_{\mathcal{H}}\colon\mathcal{H}^X\to\mathcal{H}^X$ satisfying $\kappa_{\mathcal{H}}(z\alpha) = z^{-1}\kappa_{\mathcal{H}}(\alpha)$. The involution $\kappa_{\mathcal{H}}$ coincides, along $|z|=1$, with the complex conjugation of sections with respect to the real subbundle. For a nice choice of real structures, we expect the following properties:

$$\mathbb{F}_\tau\oplus z^{-1}\kappa_{\mathcal{H}}(\mathbb{F}_\tau) = \mathcal{H}^X, \qquad \text{(Hodge Decomposition)} \tag{3}$$

$$(\kappa_{\mathcal{H}}(\alpha),\alpha)_{\mathcal{H}^X}>0, \quad \alpha\in\mathbb{F}_\tau\cap\kappa_{\mathcal{H}}(\mathbb{F}_\tau)\setminus\{0\} \quad \text{(Bilinear Inequality)} \tag{4}$$

These properties (1), (2), (3), (4) of $\frac{\infty}{2}$VHS actually reduce to the corresponding properties of a finite dimensional VHS in the conformal limit (see Section 6.2). We call the properties (3) and (4) *pure* and *polarized* respectively. First we show that (3), (4) indeed hold near the "large radius limit" *i.e.* $\tau=-x\omega$, $\Re(x)\to\infty$ for some Kähler class $\omega$, under reasonable assumptions on the real structures:

**Theorem 1.1** (Theorem 3.7). *Assume that a real structure is invariant under the monodromy (Galois) transformations given by $G^{\mathcal{H}}(\xi),\xi\in H^2(X,\mathbb{Z})$ (see Equation (31) and Proposition 3.5). If the condition (41) (which is empty when $X$ is a manifold) holds, $\mathbb{F}_\tau$ is pure (3) near the large radius limit. If moreover the condition (43) holds and $H^*(X)=\bigoplus_p H^{p,p}(X)$, $\mathbb{F}_\tau$ is polarized (4) near the large radius limit.*

In the theorem above, we allow $X$ to be an orbifold or a smooth Deligne-Mumford stack (see Theorem 3.7 for a more precise statement). Given a nice real structure

satisfying (3) and (4), quantum cohomology will be endowed with *$tt^*$-geometry* due to Cecotti-Vafa [15, 18], which has also been developed by Dubrovin [30] and Hertling [38]. The family $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ of finite dimensional Hermitian vector spaces (canonically identified with quantum cohomology) is equipped with a rich geometric structure, called Cecotti-Vafa structure (Proposition 2.12). $tt^*$-geometry also gives an example of a harmonic bundle or a twistor structure of Simpson [65]. Closely related results have been shown in a more abstract setting for TERP structures in [38, 39] and the proof of Theorem 1.1 looks similar to them. In fact, when $X$ is Fano and the Kähler class $\omega$ is $c_1(X)$, the conclusions of Theorem 1.1 can be deduced from [39, Theorem 7.3].

**Integral structures for toric orbifolds.** In the case of toric orbifolds, we concretely calculate the integral structures in quantum cohomology corresponding to those in the mirrors. A mirror partner of a toric orbifold is given by the Landau-Ginzburg model, which consists of a family $\{Y_q\}_{q\in\mathcal{M}}$ of algebraic tori and Laurent polynomials $W_q\colon Y_q \to \mathbb{C}$. In Section 4, we construct B-model $\frac{\infty}{2}$VHS from the singularity defined by the Landau-Ginzburg model. This is underlain by a canonical integral local system formed by the relative cohomology groups $R_{\mathbb{Z},(q,z)} = H^n(Y_q, \{\Re(W_q/z) \ll 0\}, \mathbb{Z})$. Assuming mirror symmetry for toric orbifolds — which will be shown in a forthcoming paper [22] — we show the following:

**Theorem 1.2** (Theorem 4.17)**.** *Let $\mathcal{X}$ be a weak Fano toric orbifold given by initial data satisfying the condition $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ in Section 4.1.4. Assume that mirror symmetry conjecture in Section 4.3 and the condition (A3) in Section 3.5 hold for $\mathcal{X}$. The integral structure of quantum cohomology of $\mathcal{X}$ pulled back from the Landau-Ginzburg mirror is given by the image of the $K$-group of topological orbifold vector bundles under the map $K(\mathcal{X}) \to \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{\mathcal{X}})$ (denoted by $z^{-\mu}z^{\rho}\Psi$ in the main body of the text):*

$$[V] \longmapsto z^{-\mu} z^{\rho} \frac{1}{(2\pi)^{n/2}} \widehat{\Gamma}_{\mathcal{X}} \cup (2\pi\sqrt{-1})^{\deg/2} \operatorname{inv}^* \widetilde{\mathrm{ch}}([V]).$$

*Here the image lies in the space of (multi-valued) flat sections of $(\mathsf{H}^{\mathcal{X}}, \widehat{\nabla}_{z\partial_z})$ and $\widehat{\Gamma}_{\mathcal{X}}$ is a universal characteristic class of $T\mathcal{X}$. (See Section 3.5 for the notation.) Under this map, Mukai pairing on $K(\mathcal{X})$ induces the pairing $(\cdot,\cdot)_{\mathcal{H}^{\mathcal{X}}}$ on $\mathcal{H}^{\mathcal{X}}$.*

The integral structures given in Theorem 1.2 make sense for general symplectic orbifolds. Furthermore, the real structure induced from this integral structure satisfies the conditions in Theorem 1.1, so in particular yields positive definite $tt^*$ geometry on $\bigoplus_p H^{p,p}(X)$ near the large radius limit for arbitrary $X$ (Definition-Proposition 3.16).

The connection between $K$-theory and quantum cohomology is compatible with the picture of *homological mirror symmetry*. In string theory, there are two types of D-branes — A-type and B-type — and homological mirror symmetry predicts that the category of A-type D-branes on $X$ is equivalent to the category of B-type D-branes on the mirror $X^\vee$. In our case, vector bundles on a toric orbifold $\mathcal{X}$ give B-type D-branes and Lefschetz thimbles in the Landau-Ginzburg mirror give A-type D-branes. Via oscillatory integrals, a Lefschetz thimble gives a flat section of the B-model $\frac{\infty}{2}$VHS of the Landau-Ginzburg model. Thus by homological mirror symmetry, a vector bundle

on $\mathcal{X}$ should also give a flat section of the A-model $\frac{\infty}{2}$VHS or quantum cohomology. In the context of toric varieties and GKZ system associated to it, these viewpoints have been emphasized by Borisov-Horja [11] and Hosono [43]. Borisov-Horja [11] identified the space of solutions to the GKZ system with the complexified $K$-group of a toric orbifold; A conjecture [43, Conjecture 6.3] of Hosono (stated in terms of hypergeometric functions) is compatible with the integral structure in Theorem 1.2. The key step in the proof of Theorem 1.2 is a calculation of an oscillatory integral over the special Lefschetz thimble $\Gamma_0$ formed by real points. It turns out that the Lefschetz thimble $\Gamma_0$ corresponds to the structure sheaf $\mathcal{O}_{\mathcal{X}}$ on the toric orbifold $\mathcal{X}$.

**Global study of quantum cohomology — Ruan's conjecture.** The study of integral structures will also be useful to understand the *global Kähler moduli space* where quantum cohomology is analytically continued. The present project was greatly motivated by the joint work [25] with Coates and Tseng, where we studied the crepant resolution conjecture for some toric examples. Ruan's crepant resolution conjecture states that when we have a crepant resolution $Y$ of a Gorenstein orbifold $\mathcal{X}$, quantum cohomology of $\mathcal{X}$ and $Y$ are related by analytic continuations. In the analytic continuation, some of the quantum parameters of $Y$ are conjectured to specialize to a root of unity at the large radius limit point of $\mathcal{X}$. In [25], we found in some examples that the quantum cohomology $\frac{\infty}{2}$VHS's of $\mathcal{X}$ and $Y$ are related by an analytic continuation followed by a certain symplectic transformation $\mathbb{U}\colon \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^{Y}$. Incorporating integral structures into this picture, we suggest the picture that the symplectic transformation $\mathbb{U}$ is induced from a (conjectural) isomorphism of $K$-groups (McKay correspondence) so that the following commutes:

$$\begin{tikzcd} K(\mathcal{X}) \arrow[r, "\cong"] \arrow[d, "z^{-\mu}z^{\rho}\Psi"'] & K(Y) \arrow[d, "z^{-\mu}z^{\rho}\Psi"'] \\ \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{\mathcal{X}}) \arrow[r, "\mathbb{U}"] & \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{Y}). \end{tikzcd}$$

Here the vertical maps relate the $K$-groups with the space of flat sections in quantum cohomology; for toric orbifolds, they should be the same as what is given in Theorem 1.2. The bottom map is induced from $\mathbb{U}\colon \mathcal{H}^{\mathcal{X}} = \Gamma(\mathbb{C}^*, \mathsf{H}^{\mathcal{X}}) \to \Gamma(\mathbb{C}^*, \mathsf{H}^{Y}) = \mathcal{H}^{Y}$. This picture, under certain assumptions, gives us a natural explanation for the specialization to a root of unity. We will use "integral periods" in the A-model to predict specialization values of some quantum parameters.

This paper is organized as follows. In Section 2, we introduce real and integral structures for a general graded $\frac{\infty}{2}$VHS. This section owes much to Hertling [38]. In Section 3, we study integral structures in (orbifold) quantum cohomology and prove Theorem 1.1. In Section 4, we calculate the integral structures pulled back from the mirror for toric orbifolds. In Section 5, we calculate the $tt^*$-geometry of $\mathbb{P}^1$. Our calculation recovers the physicists' result [16]. The aim here is to demonstrate that the Birkhoff factorization calculates $tt^*$-geometry perturbatively. In Section 6, we discuss the role of integral structures in Ruan's conjecture.

We remark that the convergence of the quantum cohomology is assumed throughout the paper. Also we consider only the even parity part of the cohomology, *i.e.* $H^*(X)$

means $\bigoplus_k H^{2k}(X)$. Note that the orbifold cohomology $H^*_{\rm orb}(\mathcal{X})$ (equipped with the Chen-Ruan's orbifold cup product) is denoted also by $H^*_{\rm CR}(\mathcal{X})$ in the literature.

**Notes added in v3:** Since the first version of the paper was written, Katzarkov-Kontsevich-Pantev [47] proposed a rational structure on a **nc**-Hodge structure defined by the same $\widehat{\Gamma}$-class independently, based on the calculation on quantum cohomology of $\mathbb{P}^n$. Here a **nc**-Hodge structure corresponds to a semi-infinite Hodge structure in this paper. They also imposed the condition that a rational structure is compatible with the Stokes structure [47, Definition 2.5].

The results on the $\widehat{\Gamma}$-integral structure and mirror symmetry for toric orbifolds in this paper were revised in the paper [45]. This revision does not contain the results on real structures, but contains more details on toric mirror symmetry.

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**Notation**

| | |
|---|---|
| $\sqrt{-1}$ | imaginary unit $\sqrt{-1}^2 = -1$ |
| $\mathcal{M}$ | complex analytic space |
| $\mathbb{D}_0 \subset \mathbb{C}$ | disc $\{z \in \mathbb{C}\ ;\ \lvert z\rvert \le 1\}$ |
| $\mathbb{D}_\infty \subset \mathbb{P}^1 \setminus \{0\}$ | disc $\{z \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1\ ;\ \lvert z\rvert \ge 1\}$ |
| $(-)\colon \mathcal{M}\times\mathbb{C} \to \mathcal{M}\times\mathbb{C}$ | map defined by $(\tau, z) \mapsto (\tau, -z)$ |
| $\mathcal{X}$ | smooth Deligne-Mumford stack |
| $I\mathcal{X}$ | inertia stack of $\mathcal{X}$ |
| $\mathsf{T} = \{0\} \cup \mathsf{T}'$ | index set of inertia components; |
| $\mathrm{inv}\colon I\mathcal{X} \to I\mathcal{X},\ \mathsf{T} \to \mathsf{T}$ | involution $(x, g) \mapsto (x, g^{-1})$ |
| $\iota_v$ | age of inertia component $v \in \mathsf{T}$ |
| $n$, $n_v$ | $\dim_{\mathbb{C}} \mathcal{X}$, $\dim_{\mathbb{C}} \mathcal{X}_v$ |
| $\mathbb{C}\{z, z^{-1}\}$, $\mathbb{C}\{z\}$, $\mathbb{C}\{z^{-1}\}$ | the space of holomorphic functions on $\mathbb{C}^*$, $\mathbb{C}$, $\mathbb{P}^1 \setminus \{0\}$. |

## 2. Real and integral structures on $\frac{\infty}{2}$VHS

We introduce real and integral structures for a semi-infinite variation of Hodge structures or $\frac{\infty}{2}$VHS. We explain that a $\frac{\infty}{2}$VHS with a real structure produces a Cecotti-Vafa structure if it is pure. A $\frac{\infty}{2}$VHS was originally introduced by Barannikov [7, 8]. A $\frac{\infty}{2}$VHS with a real structure considered here corresponds to the TERP structure due to Hertling [38] (see Remark 2.3). The exposition here largely follows the line of [38, 25].

2.1. **Definition.** Let $\mathcal{M}$ be a smooth complex analytic space. Let $\mathcal{O}_{\mathcal{M}}$ be the analytic structure sheaf on $\mathcal{M}$. Let $\mathcal{O}_{\mathcal{M}}\{z\} := \pi_*(\mathcal{O}_{\mathcal{M}\times\mathbb{C}})$ be the push-forward of $\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$ by the

projection $\pi\colon \mathcal{M}\times\mathbb{C}\to\mathcal{M}$. Here $z$ is a co-ordinate on the $\mathbb{C}$ factor. Similarly, we set $\mathcal{O}_{\mathcal{M}}\{z^{-1}\} := \pi_*(\mathcal{O}_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})$, $\mathcal{O}_{\mathcal{M}}\{z,z^{-1}\} := \pi_*(\mathcal{O}_{\mathcal{M}\times\mathbb{C}^*})$. Let $\mathbb{C}\{z\}$, $\mathbb{C}\{z,z^{-1}\}$ and $\mathbb{C}\{z^{-1}\}$ be the space of holomorphic functions on $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{P}^1\setminus\{0\}$ respectively. Let $\Omega^1_{\mathcal{M}}$ be the sheaf of holomorphic 1-forms on $\mathcal{M}$ and $\Theta_{\mathcal{M}}$ be the sheaf of holomorphic tangent vector fields on $\mathcal{M}$.

**Definition 2.1** ([25])**.** A *semi-infinite variation of Hodge structures*, or $\frac{\infty}{2}$VHS is a locally free $\mathcal{O}_{\mathcal{M}}\{z\}$-module $\mathcal{F}$ of rank $N$ endowed with a holomorphic flat connection

$$\nabla\colon \mathcal{F}\to z^{-1}\mathcal{F}\otimes\Omega^1_{\mathcal{M}}$$

and a perfect pairing

$$(\cdot,\cdot)_{\mathcal{F}}\colon \mathcal{F}\times\mathcal{F}\to\mathcal{O}_{\mathcal{M}}\{z\}$$

satisfying

$$\begin{aligned}
\nabla_X(fs) &= (Xf)s + f\nabla_X s,\\
[\nabla_X,\nabla_Y]s &= \nabla_{[X,Y]}s,\\
(s_1, f(z)s_2)_{\mathcal{F}} &= (f(-z)s_1,s_2)_{\mathcal{F}} = f(z)(s_1,s_2)_{\mathcal{F}},\\
(s_1,s_2)_{\mathcal{F}} &= (s_2,s_1)_{\mathcal{F}}|_{z\to -z},\\
X(s_1,s_2)_{\mathcal{F}} &= (\nabla_X s_1,s_2)_{\mathcal{F}} + (s_1,\nabla_X s_2)_{\mathcal{F}}
\end{aligned}$$

for sections $s, s_1, s_2$ of $\mathcal{F}$, $f\in\mathcal{O}_{\mathcal{M}}\{z\}$ and $X\in\Theta_{\mathcal{M}}$. Here, $\nabla_X$ is a map from $\mathcal{F}$ to $z^{-1}\mathcal{F}$ and $z^{-1}\mathcal{F}$ is regarded as a submodule of $\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{M}}\{z\}}\mathcal{O}_{\mathcal{M}}\{z,z^{-1}\}$. The first two properties are part of the definition of a flat connection. The pairing $(\cdot,\cdot)_{\mathcal{F}}$ is perfect in the sense that it induces an isomorphism of the fiber $\mathcal{F}_\tau$ at $\tau\in\mathcal{M}$ with $\mathrm{Hom}_{\mathbb{C}\{z\}}(\mathcal{F}_\tau,\mathbb{C}\{z\})$.

A *graded $\frac{\infty}{2}$ VHS* is a $\frac{\infty}{2}$VHS $\mathcal{F}$ endowed with a $\mathbb{C}$-endomorphism $\mathsf{Gr}\colon\mathcal{F}\to\mathcal{F}$ and an Euler vector field $E\in H^0(\mathcal{M},\Theta_{\mathcal{M}})$ satisfying

$$\begin{aligned}
\mathsf{Gr}(fs_1) &= (2(z\partial_z+E)f)s_1 + f\,\mathsf{Gr}(s_1),\\
[\mathsf{Gr},\nabla_X] &= \nabla_{2[E,X]}, \quad X\in\Theta_{\mathcal{M}},\\
2(z\partial_z+E)(s_1,s_2)_{\mathcal{F}} &= (\mathsf{Gr}(s_1),s_2)_{\mathcal{F}} + (s_1,\mathsf{Gr}(s_2))_{\mathcal{F}} - 2n(s_1,s_2)_{\mathcal{F}}
\end{aligned}$$

where $n\in\mathbb{C}$.

A $\frac{\infty}{2}$VHS is a semi-infinite analogue of the usual finite dimensional VHS without a real structure. The "semi-infinite" flag $\cdots\subset z\mathcal{F}\subset\mathcal{F}\subset z^{-1}\mathcal{F}\subset z^{-2}\mathcal{F}\subset\cdots$ of $\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{M}}\{z\}}\mathcal{O}_{\mathcal{M}}\{z,z^{-1}\}$ plays the role of the Hodge filtration. The flat connection $\nabla_X$ shifts this filtration by one — this is an analogue of the Griffiths transversality.

The structure of a graded $\frac{\infty}{2}$VHS $\mathcal{F}$ can be rephrased in terms of a locally free sheaf $\mathcal{R}^{(0)}$ over $\mathcal{M}\times\mathbb{C}$ with a flat connection $\widehat{\nabla}$. Here $\mathcal{R}^{(0)}$ is a locally free $\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$-module of rank $N$ such that $\mathcal{F}=\pi_*\mathcal{R}^{(0)}$. We define the meromorphic connection $\widehat{\nabla}$ on $\mathcal{R}^{(0)}$

$$\widehat{\nabla}\colon \mathcal{R}^{(0)}\longrightarrow \frac{1}{z}\mathcal{R}^{(0)}\otimes\pi^*\Omega^1_{\mathcal{M}}\oplus\mathcal{R}^{(0)}\frac{dz}{z^2}$$

by

$$\widehat{\nabla}s := \nabla s + \left(\frac{1}{2}\,\mathsf{Gr}(s) - \nabla_E s - \frac{n}{2}s\right)\frac{dz}{z} \tag{5}$$

for a section $s$ of $\mathcal{F} = \pi_* \mathcal{R}^{(0)}$. It is easy to see that the conditions on $\mathsf{Gr}$ and $\nabla$ above imply that $\widehat{\nabla}$ is also flat. The pairing $(\cdot,\cdot)_{\mathcal{F}}$ on $\mathcal{F}$ induces a non-degenerate pairing on $\mathcal{R}^{(0)}$:

$$(\cdot,\cdot)_{\mathcal{R}^{(0)}} \colon (-)^* \mathcal{R}^{(0)} \otimes \mathcal{R}^{(0)} \to \mathcal{O}_{\mathcal{M}\times\mathbb{C}}.$$

where $(-)\colon \mathcal{M}\times\mathbb{C} \to \mathcal{M}\times\mathbb{C}$ is a map $(\tau,z)\mapsto(\tau,-z)$. This pairing is flat with respect to $\widehat{\nabla}$ on $\mathcal{R}^{(0)}$ and $(-)^*\widehat{\nabla}$ on $(-)^*\mathcal{R}^{(0)}$. Denote by $\mathcal{R}$ the restriction of $\mathcal{R}^{(0)}$ to $\mathcal{M}\times\mathbb{C}^*$. Since $\widehat{\nabla}_{z\partial_z}$ is regular outside $z=0$, $\mathcal{R}$ gives a flat vector bundle on $\mathcal{M}\times\mathbb{C}^*$. Let $R\to\mathcal{M}\times\mathbb{C}^*$ be the $\mathbb{C}$-local system underlying the flat bundle $\mathcal{R}$. This has a pairing $(\cdot,\cdot)_R\colon (-)^*R\otimes_{\mathbb{C}} R\to\mathbb{C}$ induced from $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$.

**Definition 2.2.** Let $\mathcal{F}$ be a graded $\frac{\infty}{2}$VHS with $n\in\mathbb{Z}$. A *real structure* on $\frac{\infty}{2}$VHS is a sub $\mathbb{R}$-local system $R_{\mathbb{R}}\to\mathcal{M}\times\mathbb{C}^*$ of $R$ such that $R=R_{\mathbb{R}}\oplus\sqrt{-1}R_{\mathbb{R}}$ and the pairing takes values in $\mathbb{R}$ on $R_{\mathbb{R}}$

$$(\cdot,\cdot)_R\colon (-)^*R_{\mathbb{R}}\otimes_{\mathbb{R}} R_{\mathbb{R}}\to\mathbb{R}.$$

An *integral structure* on $\frac{\infty}{2}$VHS is a sub $\mathbb{Z}$-local system $R_{\mathbb{Z}}\to\mathcal{M}\times\mathbb{C}^*$ of $R$ such that $R=R_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}$ and the pairing takes values in $\mathbb{Z}$ on $R_{\mathbb{Z}}$

$$(\cdot,\cdot)_R\colon (-)^*R_{\mathbb{Z}}\otimes R_{\mathbb{Z}}\to\mathbb{Z}$$

and is unimodular *i.e.* induces an isomorphism $R_{\mathbb{Z},(\tau,-z)}\cong\operatorname{Hom}(R_{\mathbb{Z},(\tau,z)},\mathbb{Z})$ for $(\tau,z)\in\mathcal{M}\times\mathbb{C}^*$.

**Remark 2.3.** A graded $\frac{\infty}{2}$VHS with a real structure defined here is almost equivalent to a TERP$(n)$ structure introduced by Hertling [38]. The only difference is that the flat connection $\widehat{\nabla}$ in TERP$(n)$ structure is not assumed to arise from a grading operator $\mathsf{Gr}$ and an Euler vector field $E$. Therefore, a graded $\frac{\infty}{2}$VHS gives a TERP structure, but the converse is not true in general. For the convenience of the reader, we give differences in convention between [38] and us. Let $\tilde{\nabla}$, $\tilde{R}$, $\tilde{R}_{\mathbb{R}}$, $\tilde{P}\colon \tilde{R}\otimes(-)^*\tilde{R}\to\mathbb{C}$ denote the flat connection, $\mathbb{C}$-local system, sub $\mathbb{R}$-local system and a pairing appearing in [38]. They are related to our $\widehat{\nabla}$, $R$, $R_{\mathbb{R}}$, $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$ as

$$\begin{aligned}
&\tilde{\nabla}=\widehat{\nabla}+\frac{n}{2}\frac{dz}{z},\\
&\tilde{R}=(-z)^{-\frac{n}{2}}R,\quad \tilde{R}_{\mathbb{R}}=(-z)^{-\frac{n}{2}}R_{\mathbb{R}},\\
&\tilde{P}(s_1,s_2)=z^n(s_2,s_1)_{\mathcal{R}^{(0)}}.
\end{aligned}$$

Then $\tilde{R}$ is the local system defined by $\tilde{\nabla}$, $\tilde{P}$ is $\tilde{\nabla}$-flat and

$$\tilde{P}(\tilde{R}_{\mathbb{R},(\tau,z)}\times\tilde{R}_{\mathbb{R},(\tau,-z)})=z^n\left(z^{-n/2}R_{\mathbb{R},(\tau,-z)},(-z)^{-n/2}R_{\mathbb{R},(\tau,z)}\right)_R\subset\sqrt{-1}^n\mathbb{R}.$$

## 2.2. **Semi-infinite period map.**

**Definition 2.4.** For a graded $\frac{\infty}{2}$VHS $\mathcal{F}$, the *spaces $\mathcal{H}$, $\mathcal{V}$ of multi-valued flat sections* are defined to be

$$\begin{aligned}
&\mathcal{H}:=\{s\in\Gamma(\widetilde{\mathcal{M}}\times\mathbb{C}^*,\mathcal{R})\ ;\ \nabla s=0\},\\
&\mathcal{V}:=\{s\in\Gamma((\mathcal{M}\times\mathbb{C}^*)^{\sim},\mathcal{R})\ ;\ \widehat{\nabla}s=0\},
\end{aligned}$$

where $\widetilde{\mathcal{M}}$ and $(\mathcal{M}\times\mathbb{C}^*)^\sim$ are the universal covers of $\mathcal{M}$ and $\mathcal{M}\times\mathbb{C}^*$ respectively. The space $\mathcal{H}$ is a free $\mathbb{C}\{z,z^{-1}\}$-module, where $\mathbb{C}\{z,z^{-1}\}$ is the space of entire functions on $\mathbb{C}^*$. The space $\mathcal{V}$ is a finite dimensional $\mathbb{C}$-vector space identified with the fiber of the local system $R$. The flat connection $\widehat{\nabla}$ and the pairing $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$ on $\mathcal{R}^{(0)}$ induce an operator

$$\widehat{\nabla}_{z\partial_z}\colon \mathcal{H}\to\mathcal{H}$$

and a pairing

$$(\cdot,\cdot)_{\mathcal{H}}\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}\{z,z^{-1}\}$$

satisfying

$$\begin{aligned}(f(-z)s_1,s_2)_{\mathcal{H}} &= (s_1,f(z)s_2)_{\mathcal{H}} = f(z)(s_1,s_2)_{\mathcal{H}} \quad f(z)\in\mathbb{C}\{z,z^{-1}\},\\ (s_1,s_2)_{\mathcal{H}} &= (s_2,s_1)_{\mathcal{H}}|_{z\mapsto -z}\\ z\partial_z(s_1,s_2)_{\mathcal{H}} &= (\widehat{\nabla}_{z\partial_z}s_1,s_2)_{\mathcal{H}} + (s_1,\widehat{\nabla}_{z\partial_z}s_2)_{\mathcal{H}}.\end{aligned}$$

The free $\mathbb{C}\{z,z^{-1}\}$-module $\mathcal{H}$ can be regarded as the space of global sections of a flat vector bundle $\mathsf{H}\to\mathbb{C}^*$. Then $\mathcal{V}$ can be identified with the space of multi-valued flat sections of $\mathsf{H}$. A pairing $(\cdot,\cdot)_{\mathcal{V}}\colon \mathcal{V}\otimes_{\mathbb{C}}\mathcal{V}\to\mathbb{C}$ is defined by

$$(s_1,s_2)_{\mathcal{V}} := (s_1(\tau,e^{\pi\sqrt{-1}}z),s_2(\tau,z))_R \tag{6}$$

where $s_1(\tau,e^{\pi\sqrt{-1}}z)\in\mathcal{R}_{(\tau,-z)}$ denote the parallel translation of $s_1(\tau,z)\in\mathcal{R}_{(\tau,z)}$ along the counterclockwise path $[0,1]\ni\theta\mapsto e^{\pi\sqrt{-1}\theta}z$.

A $\frac{\infty}{2}$VHS $\mathcal{F}$ on $\mathcal{M}$ defines a map from $\widetilde{\mathcal{M}}$ to the *Segal-Wilson Grassmannian* of $\mathcal{H}$. For $u\in\mathcal{F}_\tau$ at $\tau\in\widetilde{\mathcal{M}}$, there exists a unique flat section $s_u\in\mathcal{H}$ such that $s_u(\tau)=u$. This defines an embedding of a fiber $\mathcal{F}_\tau$ into $\mathcal{H}$:

$$\mathcal{J}_\tau\colon \mathcal{F}_\tau\longrightarrow\mathcal{H},\quad u\longmapsto s_u,\quad \tau\in\widetilde{\mathcal{M}}. \tag{7}$$

We call the image $\mathbb{F}_\tau\subset\mathcal{H}$ of this embedding the *semi-infinite Hodge structure*. This is a free $\mathbb{C}\{z\}$-module of rank $N$. The family $\{\mathbb{F}_\tau\subset\mathcal{H}\}_{\tau\in\widetilde{\mathcal{M}}}$ of subspaces gives the *moving subspace realization of* $\frac{\infty}{2}$ *VHS*. Fix a $\mathbb{C}\{z,z^{-1}\}$-basis $e_1,\dots,e_N$ of $\mathcal{H}$. Then the image of a local frame $s_1,\dots,s_N$ of $\mathcal{F}$ over $\mathcal{O}_{\mathcal{M}}\{z\}$ under $\mathcal{J}_\tau$ can be written as $\mathcal{J}_\tau(s_j)=\sum_{i=1}^N e_iJ_{ij}(\tau,z)$. When $z$ is restricted to $S^1=\{|z|=1\}$, the $N\times N$ matrix $(J_{ij}(\tau,z))$ defines an element of the smooth loop group $LGL_N(\mathbb{C})=C^\infty(S^1,GL_N(\mathbb{C}))$. Another choice of a local basis of $\mathcal{F}$ changes the matrix $(J_{ij}(\tau,z))$ by right multiplication by a matrix with entries in $\mathbb{C}\{z\}$. Thus the Hodge structure $\mathbb{F}_\tau$ gives a point $(J_{ij}(\tau,z))_{ij}$ in the smooth Segal-Wilson Grassmannian $\mathrm{Gr}_{\frac{\infty}{2}}(\mathcal{H}):=LGL_N(\mathbb{C})/L^+GL_N(\mathbb{C})$ [60]. Here $L^+GL_N(\mathbb{C})$ consists of smooth loops which are the boundary values of holomorphic maps $\{z\in\mathbb{C}\,;\,|z|<1\}\to GL_N(\mathbb{C})$. The map

$$\widetilde{\mathcal{M}}\ni\tau\longmapsto\mathbb{F}_\tau\in\mathrm{Gr}_{\frac{\infty}{2}}(\mathcal{H})$$

is called the *semi-infinite period map*.

**Proposition 2.5** ([25, Proposition 2.9]). *The semi-infinite period map $\tau\mapsto\mathbb{F}_\tau$ satisfies:*

(i) $X\mathbb{F}_\tau\subset z^{-1}\mathbb{F}_\tau$ *for* $X\in\Theta_{\mathcal{M}}$*;*
(ii) $(\mathbb{F}_\tau,\mathbb{F}_\tau)_{\mathcal{H}}\subset\mathbb{C}\{z\}$*;*

(iii) $(\widehat{\nabla}_{z\partial_z} + E)\mathbb{F}_\tau \subset \mathbb{F}_\tau$. *In particular,* $\widehat{\nabla}_{z\partial_z}\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau$.

*The first property* (ii) *is an analogue of Griffiths transversality and the second* (iii) *is the Hodge-Riemann bilinear relation.*

In terms of the flat vector bundle $\mathsf{H} \to \mathbb{C}^*$ above (such that $\mathcal{H} = \Gamma(\mathbb{C}^*, \mathsf{H})$), the Hodge structure $\mathbb{F}_\tau \subset \mathcal{H}$ is considered to be an extension of $\mathsf{H}$ to $\mathbb{C}$ such that the flat connection has a pole of Poincaré rank 1 at $z = 0$.

Real and integral structures on $\frac{\infty}{2}$VHS define the following subspaces $\mathcal{H}_\mathbb{R}$, $\mathcal{V}_\mathbb{R}$, $\mathcal{V}_\mathbb{Z}$:

$$
\begin{aligned}
\mathcal{H}_\mathbb{R} &:= \{s \in \mathcal{H} \; ; \; s(\tau, z) \in R_{\mathbb{R},(\tau,z)}, \; (\tau, z) \in \widetilde{\mathcal{M}} \times S^1\} \\
\mathcal{V}_\mathbb{R} &:= \{s \in \mathcal{V} \; ; \; s(\tau, z) \in R_{\mathbb{R},(\tau,z)}, \; (\tau, z) \in (\mathcal{M} \times \mathbb{C}^*)^\sim\} \\
\mathcal{V}_\mathbb{Z} &:= \{s \in \mathcal{V} \; ; \; s(\tau, z) \in R_{\mathbb{Z},(\tau,z)}, \; (\tau, z) \in (\mathcal{M} \times \mathbb{C}^*)^\sim\}
\end{aligned}
$$

Then $\mathcal{H}_\mathbb{R}$ becomes a (not necessarily free) module over a ring $C^h(S^1, \mathbb{R})$:

$$
C^h(S^1, \mathbb{R}) := \left\{ f(z) \in \mathbb{C}\{z, z^{-1}\} \; ; \; f(z) \in \mathbb{R} \text{ if } |z| = 1 \right\}.
$$

Clearly, we have $\mathbb{C}\{z, z^{-1}\} = C^h(S^1, \mathbb{R}) \oplus \sqrt{-1}C^h(S^1, \mathbb{R})$. The involution $\kappa$ on $\mathbb{C}\{z, z^{-1}\}$ corresponding to the real form $C^h(S^1, \mathbb{R})$ is given by

$$
\kappa(f)(z) = \overline{f(\gamma(z))}, \quad f(z) \in \mathbb{C}\{z, z^{-1}\},
$$

where $\gamma(z) = 1/\overline{z}$ and the $\overline{\phantom{x}}$ in the right-hand side is the complex conjugate. We also have $\mathcal{H} \cong \mathcal{H}_\mathbb{R} \oplus \sqrt{-1}\mathcal{H}_\mathbb{R}$. This real form $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ defines an involution $\kappa_\mathcal{H} \colon \mathcal{H} \to \mathcal{H}$ such that

$$
\begin{aligned}
\kappa_\mathcal{H}(fs) &= \kappa(f)\kappa_\mathcal{H}(s), \\
\kappa_\mathcal{H}\widehat{\nabla}_{z\partial_z} &= -\widehat{\nabla}_{z\partial_z}\kappa_\mathcal{H}, \\
\kappa((s_1, s_2)_\mathcal{H}) &= (\kappa_\mathcal{H}(s_1), \kappa_\mathcal{H}(s_2))_\mathcal{H}.
\end{aligned} \tag{8}
$$

Similarly, we have $\mathcal{V} = \mathcal{V}_\mathbb{R} \oplus \sqrt{-1}\mathcal{V}_\mathbb{R}$; we denote by $\kappa_\mathcal{V} \colon \mathcal{V} \to \mathcal{V}$ the involution defined by the real structure $\mathcal{V}_\mathbb{R}$.

**Remark 2.6.** In the context of the smooth Grassmannian, it is more natural to work over $C^\infty(S^1, \mathbb{C})$ instead of $\mathbb{C}\{z, z^{-1}\}$. We put

$$
\widetilde{\mathcal{H}} := \mathcal{H} \otimes_{\mathbb{C}\{z,z^{-1}\}} C^\infty(S^1, \mathbb{C}), \quad \widetilde{\mathbb{F}}_\tau := \mathbb{F}_\tau \otimes_{\mathbb{C}\{z\}} \mathcal{O}(\mathbb{D}_0),
$$

where $\mathcal{O}(\mathbb{D}_0)$ is a subspace of $C^\infty(S^1, \mathbb{C})$ consisting of functions which are the boundary values of holomorphic functions on the interior of the disc $\mathbb{D}_0 = \{z \in \mathbb{C} \; ; \; |z| \le 1\}$. The involution $\kappa_\mathcal{H} \colon \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ and the real form $\widetilde{\mathcal{H}}_\mathbb{R}$ is defined similarly and the same properties hold. Conversely, using the flat connection $\widehat{\nabla}_{z\partial_z}$ in the $z$-direction, one can recover $\mathbb{F}_\tau$ from $\widetilde{\mathbb{F}}_\tau$ since flat sections of $\widehat{\nabla}_{z\partial_z}$ determine an extension of the bundle on $\mathbb{D}_0$ to $\mathbb{C}$.

2.3. **Pure and polarized $\frac{\infty}{2}$VHS.** Following Hertling [38], we define an extension $\widehat{K}$ of $\mathcal{R}^{(0)}$ across $z = \infty$. The properties “pure and polarized” for $\mathcal{F}$ are defined in terms of this extension.

**Definition 2.7** (Extension of $\mathcal{R}^{(0)}$ across $z=\infty$)**.** Let $\gamma\colon \mathcal{M}\times\mathbb{P}^1\to\mathcal{M}\times\mathbb{P}^1$ be the map defined by $\gamma(\tau,z)=(\tau,1/\overline{z})$. Let $\overline{\mathcal{M}}$ denote the complex conjugate of $\mathcal{M}$, *i.e.* $\overline{\mathcal{M}}$ is the same as $\mathcal{M}$ as a real-analytic manifold but holomorphic functions on $\overline{\mathcal{M}}$ are anti-holomorphic functions on $\mathcal{M}$. The pull-back $\gamma^*\mathcal{R}^{(0)}$ of $\mathcal{R}^{(0)}$ has the structure of an $\mathcal{O}_{\mathcal{M}\times\overline{(\mathbb{P}^1\setminus\{0\})}}$-module. Thus its complex conjugate $\overline{\gamma^*\mathcal{R}^{(0)}}$ has the structure of an $\mathcal{O}_{\overline{\mathcal{M}}\times(\mathbb{P}^1\setminus\{0\})}$-module. Regarding $\mathcal{R}^{(0)}$ and $\overline{\gamma^*\mathcal{R}^{(0)}}$ as real-analytic vector bundles over $\mathcal{M}\times\mathbb{C}$ and $\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})$, we glue them along $\mathcal{M}\times\mathbb{C}^*$ by the fiberwise map

$$\mathcal{R}^{(0)}_{(\tau,z)}\xrightarrow{\ \kappa\ }\overline{\mathcal{R}^{(0)}}_{(\tau,z)}\xrightarrow{P(\gamma(z),z)}\overline{\mathcal{R}^{(0)}}_{(\tau,\gamma(z))}=\overline{\gamma^*\mathcal{R}^{(0)}}_{(\tau,z)},\quad z\in\mathbb{C}^*. \tag{9}$$

Here the first map $\kappa$ is the real involution on $\mathcal{R}^{(0)}_{(\tau,z)}$ with respect to the real form $R_{\mathbb{R},(\tau,z)}$ and the second map $P(\gamma(z),z)$ is the parallel translation for the flat connection $\widehat{\nabla}$ along the path $[0,1]\ni t\mapsto(1-t)z+t\gamma(z)$. Define $\widehat{K}\to\mathcal{M}\times\mathbb{P}^1$ to be the real-analytic complex vector bundle obtained by gluing $\mathcal{R}^{(0)}$ and $\overline{\gamma^*\mathcal{R}^{(0)}}$ in this way. Notice that $\widehat{K}|_{\tau\times\mathbb{P}^1}$ has the structure of a holomorphic vector bundle since the gluing map (9) preserves the holomorphic structure in the $\mathbb{P}^1$-direction.

**Definition 2.8.** A graded $\frac{\infty}{2}$VHS $\mathcal{F}$ with a real structure is called *pure* at $\tau\in\mathcal{M}$ if $\widehat{K}|_{\{\tau\}\times\mathbb{P}^1}$ is trivial as a holomorphic vector bundle on $\mathbb{P}^1$.

A pure graded $\frac{\infty}{2}$VHS with a real structure here corresponds to the (trTERP) structure in [38]. Here we follow the terminology in [39].

We rephrase the purity in terms of the moving subspace realization $\{\mathbb{F}_\tau\subset\mathcal{H}\}$. When we identify $\mathcal{H}$ with the space of global sections of $\widehat{K}|_{\{\tau\}\times\mathbb{C}^*}=\mathcal{R}|_{\{\tau\}\times\mathbb{C}^*}$, it is easy to see that the involution $\kappa_{\mathcal{H}}\colon\mathcal{H}\to\mathcal{H}$ is induced by the gluing map (9). Then $\mathbb{F}_\tau$ is identified with the space of holomorphic sections of $\widehat{K}|_{\{\tau\}\times\mathbb{C}^*}$ which can extend to $\{\tau\}\times\mathbb{C}$; $\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ is identified with the space of holomorphic sections of $\widehat{K}|_{\{\tau\}\times\mathbb{C}^*}$ which can extend to $\{\tau\}\times(\mathbb{P}^1\setminus\{0\})$. Similarly, $\widetilde{\mathbb{F}}_\tau$ (resp. $\kappa_{\mathcal{H}}(\widetilde{\mathbb{F}}_\tau)$) is identified with the space of smooth sections of $\widehat{K}|_{\{\tau\}\times S^1}$ which can extend to holomorphic sections on $\mathbb{D}_0$ (resp. $\mathbb{D}_\infty$), where $\mathbb{D}_0=\{z\in\mathbb{C}\ ;\ |z|\le 1\}$, $\mathbb{D}_\infty=\{z\in\mathbb{C}\cup\{\infty\}=\mathbb{P}^1\ ;\ |z|\ge 1\}$ and $\widetilde{\mathbb{F}}_\tau$ is the space in Remark 2.6.

**Proposition 2.9.** *A graded $\frac{\infty}{2}$VHS $\mathcal{F}$ with a real structure is pure at $\tau\in\mathcal{M}$ if and only if one of the following natural maps is an isomorphism:*

$$\mathbb{F}_\tau\cap\kappa_{\mathcal{H}}(\mathbb{F}_\tau)\longrightarrow\mathbb{F}_\tau/z\mathbb{F}_\tau, \tag{10}$$

$$(\mathbb{F}_\tau\cap\mathcal{H}_{\mathbb{R}})\otimes\mathbb{C}\longrightarrow\mathbb{F}_\tau/z\mathbb{F}_\tau, \tag{11}$$

$$\mathbb{F}_\tau\oplus z^{-1}\kappa_{\mathcal{H}}(\mathbb{F}_\tau)\longrightarrow\mathcal{H}. \tag{12}$$

*This holds also true when $\mathbb{F}_\tau$, $\mathcal{H}$, $\mathcal{H}_{\mathbb{R}}$ are replaced with $\widetilde{\mathbb{F}}_\tau$, $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{H}}_{\mathbb{R}}$ in Remark 2.6. When $\mathcal{F}$ is pure at some $\tau$, $\mathcal{H}_{\mathbb{R}}$ is a free module over $C^h(S^1,\mathbb{R})$.*

*Proof.* Under the identifications we explained above, $\mathbb{F}_\tau\cap\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ is identified with the space of global sections of $\widehat{K}|_{\{\tau\}\times\mathbb{P}^1}$ and the natural map $\mathbb{F}_\tau\cap\kappa_{\mathcal{H}}(\mathbb{F}_\tau)\to\mathbb{F}_\tau/z\mathbb{F}_\tau$ corresponds to the restriction to $z=0$ (note that $\mathbb{F}_\tau/z\mathbb{F}_\tau\cong\widehat{K}_{(\tau,0)}$). Therefore (10) is an isomorphism if and only if $K|_{\{\tau\}\times\mathbb{P}^1}$ is trivial. $\mathbb{F}_\tau\cap\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ is invariant under $\kappa_{\mathcal{H}}$

and its real form is given by $\mathbb{F}_\tau \cap \mathcal{H}_\mathbb{R}$. Therefore, we have $\mathbb{F}_\tau \cap \kappa_\mathcal{H}(\mathbb{F}_\tau) \cong (\mathbb{F}_\tau \cap \mathcal{H}_\mathbb{R}) \otimes \mathbb{C}$. Thus (10) is an isomorphism if and only if so is (11). Similarly, we can see that (12) is an isomorphism if $\widehat{K}|_{\{\tau\}\times\mathbb{P}^1}$ is trivial. Conversely, we show that (10) is an isomorphism if so is (12). The injectivity of the map $\mathbb{F}_\tau \cap \kappa_\mathcal{H}(\mathbb{F}_\tau) \to \mathbb{F}_\tau/z\mathbb{F}_\tau$ is easy to check. Take $v \in \mathbb{F}_\tau$. By assumption, $z^{-1}v = v_1 + v_2$ for some $v_1 \in \mathbb{F}_\tau$ and $v_2 \in z^{-1}\kappa_\mathcal{H}(\mathbb{F}_\tau)$. Thus $v - zv_1 = zv_2 \in \mathbb{F}_\tau \cap \kappa_\mathcal{H}(\mathbb{F}_\tau)$ and the image of this element in $\mathbb{F}_\tau/z\mathbb{F}_\tau$ is $[v]$. The discussion on the spaces $\widetilde{\mathbb{F}}_\tau$, $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_\mathbb{R}$ are similar.

The last statement: Since $\mathbb{F}_\tau \cap \kappa_\mathcal{H}(\mathbb{F}_\tau) \cong (\mathbb{F}_\tau \cap \mathcal{H}_\mathbb{R}) \otimes \mathbb{C}$, we can take a global basis of the trivial bundle $\widehat{K}|_{\{\tau\}\times\mathbb{P}^1}$ from $\mathbb{F}_\tau \cap \mathcal{H}_\mathbb{R}$. The module $\mathcal{H}_\mathbb{R}$ is freely generated by such a basis over $C^h(S^1, \mathbb{R})$. $\square$

**Definition 2.10.** A graded $\frac{\infty}{2}$VHS $\mathcal{F}$ with a real structure is called *polarized* at $\tau \in \mathcal{M}$ if the Hermitian pairing $h$ on $\mathbb{F}_\tau \cap \kappa_\mathcal{H}(\mathbb{F}_\tau) \cong \Gamma(\mathbb{P}^1, \widehat{K}|_{\{\tau\}\times\mathbb{P}^1})$ defined by

$$h\colon s_1 \times s_2 \longmapsto (\kappa_\mathcal{H}(s_1), s_2)_\mathcal{H}$$

is positive definite. Note that this pairing takes values in $\mathbb{C}$ since $(\mathbb{F}_\tau, \mathbb{F}_\tau)_\mathcal{H} \subset \mathbb{C}\{z\}$ and $(\kappa_\mathcal{H}(\mathbb{F}_\tau), \kappa_\mathcal{H}(\mathbb{F}_\tau))_\mathcal{H} \subset \mathbb{C}\{z^{-1}\}$ by (8). It is easy to show that a polarized $\frac{\infty}{2}$VHS is necessarily pure at the same point.

**Remark 2.11.** In order to obtain a basis of $\widetilde{\mathbb{F}}_\tau \cap \kappa_\mathcal{H}(\widetilde{\mathbb{F}}_\tau)$ or $\widetilde{\mathbb{F}}_\tau \cap \widetilde{\mathcal{H}}_\mathbb{R}$, we can make use of Birkhoff or Iwasawa factorization. Take an $\mathcal{O}(\mathbb{D}_0)$-basis $s_1, \dots, s_N$ of $\widetilde{\mathbb{F}}_\tau$. Define an element $A(z) = (A_{ij}(z))$ of the loop group $LGL_N(\mathbb{C})$ by

$$[\kappa_\mathcal{H}(s_1), \dots, \kappa_\mathcal{H}(s_N)] = [s_1, \dots, s_N]A(z), \quad \textit{i.e.}\ \kappa_\mathcal{H}(s_i) = \sum_j s_j A_{ji}(z).$$

If $A(z)$ admits the Birkhoff factorization $A(z) = B(z)C(z)$, where $B(z)$ and $C(z)$ are holomorphic maps $B(z)\colon \mathbb{D}_0 \to GL_N(\mathbb{C})$, $C(z)\colon \mathbb{D}_\infty \to GL_N(\mathbb{C})$ such that $B(0) = \mathbf{1}$, then we obtain a $\mathbb{C}$-basis of $\widetilde{\mathbb{F}}_\tau \cap \kappa_\mathcal{H}(\widetilde{\mathbb{F}}_\tau)$ as

$$[\kappa_\mathcal{H}(s_1), \dots, \kappa_\mathcal{H}(s_N)]C(z)^{-1} = [s_1, \dots, s_N]B(z). \tag{13}$$

Here, $\mathcal{F}$ is pure at $\tau \in \mathcal{M}$ if and only if $A(z)$ admits the Birkhoff factorization, *i.e.* $A(z)$ is in the "big cell" of the loop group. In particular, the purity is an open condition for $\tau \in \mathcal{M}$. On the other hand, the Iwasawa-type factorization appears as follows. Assume that we have a basis $e_1, \dots, e_N$ of $\widetilde{\mathcal{H}}_\mathbb{R}$ over $C^\infty(S^1, \mathbb{R})$ such that $(e_i, e_j)_{\widetilde{\mathcal{H}}} = \delta_{ij}$ and a basis $s_1, \dots, s_N$ of $\widetilde{\mathbb{F}}_\tau$ over $\mathcal{O}(\mathbb{D}_0)$ such that $(s_i, s_j)_{\widetilde{\mathcal{H}}} = \delta_{ij}$. Define a matrix $J(z)$ by

$$[s_1, \dots, s_N] = [e_1, \dots, e_N]J(z).$$

This $J(z)$ lies in the *twisted loop group* $LGL_N(\mathbb{C})_{\mathrm{tw}}$:

$$LGL_N(\mathbb{C})_{\mathrm{tw}} := \{J\colon S^1 \to GL_N(\mathbb{C})\ ;\ J(-z)^{\mathrm{T}}J(z) = \mathbf{1}\}.$$

If $J(z)$ admits an Iwasawa-type factorization $J(z) = U(z)B(z)$, where $U\colon S^1 \to GL_N(\mathbb{R})$ with $U(-z)^{\mathrm{T}}U(z) = \mathbf{1}$ and $B\colon \mathbb{D}_0 \to GL_N(\mathbb{C})$ with $B(-z)^{\mathrm{T}}B(z) = \mathbf{1}$, then we obtain an $\mathbb{R}$-basis of $\widetilde{\mathbb{F}}_\tau \cap \widetilde{\mathcal{H}}_\mathbb{R}$ as

$$[s_1, \dots, s_N]B(z)^{-1} = [e_1, \dots, e_N]U(z)$$

which is orthonormal with respect to $(\cdot,\cdot)_{\widetilde{\mathcal{H}}}$. In this case, the pairing $(\cdot,\cdot)_{\widetilde{\mathcal{H}}}$ restricted to $\widetilde{\mathbb{F}}_\tau \cap \widetilde{\mathcal{H}}_{\mathbb{R}}$ is an $\mathbb{R}$-valued *positive definite* symmetric form. The map $\tau \mapsto J(z)$ gives rise to the semi-infinite period map in Section 2.2:

$$\mathcal{M} \ni \tau \longmapsto [J(z)] \in LGL_N(\mathbb{C})_{\mathrm{tw}}/LGL_N^+(\mathbb{C})_{\mathrm{tw}}.$$

Here, $\mathcal{F}$ is pure at $\tau$ and $(\widetilde{\mathbb{F}}_\tau \cap \widetilde{\mathcal{H}}_{\mathbb{R}}, (\cdot,\cdot)_{\widetilde{\mathcal{H}}})$ is positive definite if and only if the image of this map lies in the $LGL_N(\mathbb{R})_{\mathrm{tw}}$-orbit of $[\mathbf{1}]$. This orbit is open, but not dense. We owe the Lie group theoretic viewpoint here to Guest [36, 37].

2.4. **Cecotti-Vafa structure.** We describe the Cecotti-Vafa structure ($tt^*$-geometry) associated to a pure graded $\frac{\infty}{2}$VHS with a real structure.

Define a complex vector bundle $K \to \mathcal{M}$ by $K := \widehat{K}|_{\mathcal{M}\times\{0\}}$. This is the real analytic vector bundle underlying $\mathcal{F}/z\mathcal{F} \cong \mathcal{R}^{(0)}|_{\mathcal{M}\times\{0\}}$. Let $\mathcal{A}^p_{\mathcal{M}}$ be the sheaf of complex-valued $C^\infty$ $p$-forms on $\mathcal{M}$ and $\mathcal{A}^1_{\mathcal{M}} = \mathcal{A}^{1,0}_{\mathcal{M}} \oplus \mathcal{A}^{0,1}_{\mathcal{M}}$ be the type decomposition.

**Proposition 2.12** ([38, Theorem 2.19])**.** *Assume that a graded $\frac{\infty}{2}$VHS $\mathcal{F}$ with a real structure is pure over $\mathcal{M}$. Then the vector bundle $K$ is equipped with a Cecotti-Vafa structure $(\kappa, g, C, \widetilde{C}, D, \mathcal{Q}, \mathcal{U}, \overline{\mathcal{U}})$. This is given by the data (see (14), (15), (16), (17)):*

- *A complex-antilinear involution $\kappa\colon K_\tau \to K_\tau$;*
- *A non-degenerate, symmetric, $\mathbb{C}$-bilinear metric $g\colon K_\tau \times K_\tau \to \mathbb{C}$ which is real with respect to $\kappa$, i.e. $g(\kappa u_1, \kappa u_2) = \overline{g(u_1,u_2)}$;*
- *Endomorphisms $C \in \mathrm{End}(K)\otimes \mathcal{A}^{1,0}_{\mathcal{M}}$, $\widetilde{C} \in \mathrm{End}(K)\otimes\mathcal{A}^{0,1}_{\mathcal{M}}$ such that $\widetilde{C}_{\overline{\imath}} = \kappa C_i \kappa$;*
- *A connection $D\colon K \to K \otimes \mathcal{A}^1_{\mathcal{M}}$ real with respect to $\kappa$, i.e. $D_{\overline{\imath}} = \kappa D_i \kappa$;*
- *Endomorphisms $\mathcal{Q}, \mathcal{U}, \overline{\mathcal{U}} \in \mathrm{End}(K)$ such that $\mathcal{U} = C_E$, $\overline{\mathcal{U}} = \kappa\mathcal{U}\kappa = \widetilde{C}_{\overline{E}}$ and $\mathcal{Q}\kappa = -\kappa\mathcal{Q}$*

*satisfying the integrability conditions*

$$\begin{aligned}
&[D_i, D_j] = 0, \quad D_iC_j - D_jC_i = 0, \quad [C_i, C_j] = 0,\\
&[D_{\overline{\imath}}, D_{\overline{\jmath}}] = 0, \quad D_{\overline{\imath}}\widetilde{C}_{\overline{\jmath}} - D_{\overline{\jmath}}\widetilde{C}_{\overline{\imath}} = 0, \quad [\widetilde{C}_{\overline{\imath}}, \widetilde{C}_{\overline{\jmath}}] = 0,\\
&D_i\widetilde{C}_{\overline{\jmath}} = 0, \quad D_{\overline{\imath}}C_j = 0, \quad [D_i, D_{\overline{\jmath}}] + [C_i, \widetilde{C}_{\overline{\jmath}}] = 0,\\
&D_i\overline{\mathcal{U}} = 0, \quad D_i\mathcal{Q} - [\overline{\mathcal{U}}, C_i] = 0, \quad D_i\mathcal{U} - C_i + [\mathcal{Q}, C_i] = 0, \quad [\mathcal{U}, C_i] = 0,\\
&D_{\overline{\imath}}\mathcal{U} = 0, \quad D_{\overline{\imath}}\mathcal{Q} + [\mathcal{U}, \widetilde{C}_{\overline{\imath}}] = 0, \quad D_{\overline{\imath}}\overline{\mathcal{U}} - \widetilde{C}_{\overline{\imath}} - [\mathcal{Q}, \widetilde{C}_{\overline{\imath}}] = 0, \quad [\overline{\mathcal{U}}, \widetilde{C}_{\overline{\imath}}] = 0,
\end{aligned}$$

*and the compatibility with the metric*

$$\begin{aligned}
&\partial_i g(u_1, u_2) = g(D_iu_1, u_2) + g(u_1, D_iu_2),\\
&\partial_{\overline{\imath}} g(u_1, u_2) = g(D_{\overline{\imath}}u_1, u_2) + g(u_1, D_{\overline{\imath}}u_2),\\
&g(C_iu_1, u_2) = g(u_1, C_iu_2), \quad g(\widetilde{C}_{\overline{\imath}}u_1, u_2) = g(u_1, \widetilde{C}_{\overline{\imath}}u_2),\\
&g(\mathcal{U}u_1, u_2) = g(u_1, \mathcal{U}u_2), \quad g(\overline{\mathcal{U}}u_1, u_2) = g(u_1, \overline{\mathcal{U}}u_2),\\
&g(\mathcal{Q}u_1, u_2) + g(u_1, \mathcal{Q}u_2) = 0.
\end{aligned}$$

*Here we chose a local complex co-ordinate system $\{t^i\}$ on $\mathcal{M}$ and used the notation $D_i = D_{\partial/\partial t^i}$, $D_{\overline{\imath}} = D_{\partial/\partial\overline{t^i}}$, etc.*

A concrete example of the Cecotti-Vafa structure will be given in Section 5. We explain the construction of the above data from the $\frac{\infty}{2}$VHS $\mathcal{F}$. Because $\mathcal{F}$ is pure, we have a canonical identification

$$K_\tau \cong \Gamma(\mathbb{P}^1, \widehat{K}|_{\{\tau\}\times\mathbb{P}^1}) \cong \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau).$$

The involution $\kappa_{\mathcal{H}}$ and the pairing $(\cdot,\cdot)_{\mathcal{H}}$ restricted to $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ induce an involution $\kappa$ and a $\mathbb{C}$-bilinear pairing $g$ on $K_\tau$:

$$\Phi_\tau(\kappa(u)) := \kappa_{\mathcal{H}}(\Phi_\tau(u)), \tag{14}$$

$$g(u_1, u_2) := (\Phi_\tau(u_1), \Phi_\tau(u_2))_{\mathcal{H}} \in \mathbb{C} \tag{15}$$

satisfying

$$g(\kappa u_1, \kappa u_2) = \overline{g(u_1, u_2)}, \quad g(u_1, u_2) = g(u_2, u_1).$$

Note that the subspace $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ depends on the parameter $\tau$ real analytically. A $C^\infty$-version of the Griffiths transversality gives

$$X^{(1,0)}\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau, \qquad X^{(0,1)}\mathbb{F}_\tau \subset \mathbb{F}_\tau,$$
$$X^{(1,0)}\kappa_{\mathcal{H}}(\mathbb{F}_\tau) \subset \kappa_{\mathcal{H}}(\mathbb{F}_\tau), \qquad X^{(0,1)}\kappa_{\mathcal{H}}(\mathbb{F}_\tau) \subset z\kappa_{\mathcal{H}}(\mathbb{F}_\tau),$$

where $X^{(1,0)} \in T^{1,0}\mathcal{M}$ and $X^{(0,1)} \in T^{0,1}\mathcal{M}$. For $X^{(1,0)} \in T^{1,0}_\tau\mathcal{M}$, we have

$$X^{(1,0)}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \subset z^{-1}\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) = z^{-1}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \oplus (\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)).$$

Similarly for $X^{(0,1)} \in T^{(0,1)}_\tau\mathcal{M}$, we have

$$X^{(0,1)}(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \subset (\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)) \oplus z(\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)).$$

Hence we can define endomorphisms $C\colon K \to K \otimes \mathcal{A}^{1,0}$, $\widetilde{C}\colon K \to K \otimes \mathcal{A}^{0,1}$, and a connection $D\colon K \to K \otimes \mathcal{A}^1$ by

$$X\Phi_\tau(u_\tau) = z^{-1}\Phi_\tau(C_X(u_\tau)) + \Phi_\tau(D_X(u_\tau)) + z\Phi_\tau(\widetilde{C}_X(u_\tau)) \tag{16}$$

for a section $u_\tau$ of $K$. By applying $\kappa_{\mathcal{H}}$ on the both hand sides,

$$\overline{X}\Phi_\tau(\kappa u_\tau) = z^{-1}\Phi_\tau(\kappa\widetilde{C}_X(u_\tau)) + \Phi_\tau(\kappa D_X(u_\tau)) + z\Phi_\tau(\kappa C_X(u_\tau)).$$

Therefore, we must have

$$C_{\overline{X}}\kappa = \kappa\widetilde{C}_X, \quad \kappa D_X = D_{\overline{X}}\kappa, \quad X \in T\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}.$$

Similarly, we can define endomorphisms $\mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q}\colon K \to K$ by

$$\widehat{\nabla}_{z\partial_z}\Phi_\tau(u_\tau) = -z^{-1}\Phi_\tau(\mathcal{U}(u_\tau)) + \Phi_\tau(\mathcal{Q}(u_\tau)) + z\Phi_\tau(\overline{\mathcal{U}}(u_\tau)), \tag{17}$$

Because $\widehat{\nabla}_{z\partial_z}$ is purely imaginary (8), we have

$$\kappa\mathcal{Q} = -\mathcal{Q}\kappa, \quad \overline{\mathcal{U}} = \kappa\mathcal{U}\kappa.$$

By $(\widehat{\nabla}_{z\partial_z} + E)\mathbb{F}_\tau \subset \mathbb{F}_\tau$ in Proposition 2.5, we find

$$\mathcal{U} = C_E, \quad \overline{\mathcal{U}} = \widetilde{C}_{\overline{E}}.$$

We have a canonical isomorphism

$$\pi^*K \cong \widehat{K}, \quad \text{where } \pi\colon \mathcal{M} \times \mathbb{P}^1 \to \mathcal{M}.$$

Let $C^{\infty h}(\pi^* K)$ be the sheaf of $C^\infty$ sections of $\pi^* K \cong \widehat{K}$ which are holomorphic on each fiber $\{\tau\} \times \mathbb{P}^1$. Under the isomorphism above, the flat connection $\widehat{\nabla}$ on $\mathcal{R}^{(0)} = \widehat{K}|_{\mathcal{M}\times\mathbb{C}}$ can be written in the form:

$$\begin{aligned}\widehat{\nabla}\colon C^{\infty h}(\pi^* K) \longrightarrow C^{\infty h}(\pi^* K) \otimes \Big( & z^{-1}\mathcal{A}^{1,0}_{\mathcal{M}} \oplus \mathcal{A}^1_{\mathcal{M}} \oplus z\mathcal{A}^{0,1}_{\mathcal{M}} \\ & \oplus (z^{-1}\mathcal{A}^0_{\mathcal{M}} \oplus \mathcal{A}^0_{\mathcal{M}} \oplus z\mathcal{A}^0_{\mathcal{M}})\frac{dz}{z}\Big)\end{aligned}$$

$$\widehat{\nabla} = z^{-1}C + D + z\widetilde{C} + (z\partial_z - z^{-1}\mathcal{U} + \mathcal{Q} + z\overline{\mathcal{U}}) \otimes \frac{dz}{z}. \tag{18}$$

Under the same isomorphism, the pairing $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$ on $\mathcal{R}^{(0)} = \widehat{K}|_{\mathcal{M}\times\mathbb{C}}$ can be written as

$$\begin{aligned}C^{\infty h}((-)^*(\pi^* K)) \otimes C^{\infty h}(\pi^* K) &\to C^{\infty h}(\mathcal{M} \times \mathbb{P}^1) \\ s_1(\tau,-z) \otimes s_2(\tau,z) &\longmapsto g(s_1(\tau,-z), s_2(\tau,z)).\end{aligned}$$

Unpacking the flatness of $\widehat{\nabla}$ and $\widehat{\nabla}$-flatness of the pairing in terms of $C, \widetilde{C}, D, \mathcal{U}, \mathcal{Q}$ and $g$, we arrive at the equations in Proposition 2.12. Note that the pairing $h$ in Definition 2.10 gives a Hermitian metric on $K$ and is related to $g$ by

$$h(u_1, u_2) = g(\kappa(u_1), u_2).$$

**Remark 2.13.** (i) The $(0,1)$-part $\widehat{\nabla}_{\bar{i}} = D_{\bar{i}} + z\widetilde{C}_{\bar{i}}$ of the flat connection (18) gives the holomorphic structure on $\widehat{K}|_{\mathcal{M}\times\{z\}}$ which corresponds to the holomorphic structure on $\mathcal{R}^{(0)}$. In particular, $D$ is identified with the canonical connection associated to the Hermitian metric $h$ on the holomorphic vector bundle $\mathcal{F}/z\mathcal{F}$. Similarly, the $(1,0)$-part $D_i + z^{-1}C_i$ gives an anti-holomorphic structure on $\widehat{K}|_{\mathcal{M}\times\{z\}}$ which corresponds to the anti-holomorphic structure on $\overline{\gamma^*\mathcal{R}^{(0)}}$.

(ii) Among the data of the Cecotti-Vafa structure, one can define the data $(C$, $D_E + \mathcal{Q}$, $\mathcal{U}$, $g)$ without choosing a real structure. In fact, $C_X$ is given by the map $\mathcal{F}/z\mathcal{F} \ni [s] \mapsto [z\nabla_X s] \in \mathcal{F}/z\mathcal{F}$, $D_E + \mathcal{Q}$ is given by the map $\mathcal{F}/z\mathcal{F} \ni [s] \mapsto [\frac{1}{2}(\mathsf{Gr} - n)s] \in \mathcal{F}/z\mathcal{F}$, $\mathcal{U} = C_E$, and $g$ is given by $g([s_1],[s_2]) = (s_1,s_2)_{\mathcal{F}}|_{z=0}$ for $s_i \in \mathcal{F}$. In the case of quantum cohomology, $C_i$ is the quantum multiplication $\phi_i\circ$ by some $\phi_i \in H^*_{\rm orb}(\mathcal{X})$ (see (20), (22)) and $g$ is the Poincaré pairing.

**Remark 2.14.** A Frobenius manifold structure on $\mathcal{M}$ arises from a miniversal $\frac{\infty}{2}$VHS (in the sense of [25, Definition 2.8]) without a real structure. To obtain a Frobenius manifold structure, we need a choice of an opposite subspace $\mathcal{H}_- \subset \mathcal{H}$: a sub free $\mathbb{C}\{z^{-1}\}$-module $\mathcal{H}_-$ of $\mathcal{H}$ satisfying

$$\mathcal{H} = \mathbb{F}_\tau \oplus \mathcal{H}_-, \quad \widehat{\nabla}_{z\partial_z}\mathcal{H}_- \subset \mathcal{H}_-.$$

The choice of $\mathcal{H}_-$ corresponds to giving a logarithmic extension of the flat bundle $(\mathcal{H}, \widehat{\nabla}_{z\partial_z})$ at $z = \infty$. A graded $\frac{\infty}{2}$VHS with the choice of an opposite subspace corresponds to the (trTLEP)-structure in Hertling [38]. See [7, 38, 25] for the construction of Frobenius manifolds from this viewpoint. In the $tt^*$-geometry, the complex conjugate $\kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ of the Hodge structure $\mathbb{F}_\tau$ plays the role of the opposite subspace (see (12)). When a miniversal $\frac{\infty}{2}$VHS is equipped with both a real structure and an opposite subspace, under certain conditions, $\mathcal{M}$ has a CDV (Cecotti-Dubrovin-Vafa) structure,

which dominates both Frobenius manifold structure and Cecotti-Vafa structure on $T\mathcal{M}$. See [38, Theorem 5.15] for more details.

## 3. Real and integral structures on the A-model

In this section, we study real and integral structures in orbifold quantum cohomology. The quantum cohomology and Gromov-Witten theory for orbifolds have been developed by Chen-Ruan [20] in the symplectic category and by Abramovich-Graber-Vistoli [2] in the algebraic category. The definition of real and integral structures makes sense for both categories, but we will work in the algebraic category. In the proof of Theorem 3.7, we will need Lefschetz decomposition which may not hold in the symplectic category. Also, we only consider the even parity part of the cohomology group.

3.1. **Orbifold quantum cohomology.** Let $\mathcal{X}$ be a proper smooth Deligne-Mumford stack over $\mathbb{C}$. Let $I\mathcal{X}$ denote the *inertia stack* of $\mathcal{X}$, defined by the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ of the two diagonal morphisms $\Delta\colon \mathcal{X} \to \mathcal{X}\times\mathcal{X}$. An object of $I\mathcal{X}$ is given by a pair $(x,g)$ of a point $x\in\mathcal{X}$ and $g\in \mathrm{Aut}(x)$. We call $g$ a *stabilizer* of $(x,g)\in I\mathcal{X}$. Let $\mathsf{T}$ be the index set of components of the $I\mathcal{X}$. Let $0\in\mathsf{T}$ be the distinguished element corresponding to the trivial stabilizer. Set $\mathsf{T}'=\mathsf{T}\setminus\{0\}$. We have

$$I\mathcal{X} = \bigsqcup_{v\in\mathsf{T}} \mathcal{X}_v = \mathcal{X}_0 \cup \bigsqcup_{v\in\mathsf{T}'} \mathcal{X}_v, \quad \mathcal{X}_0 = \mathcal{X}.$$

For each connected component $\mathcal{X}_v$ of $I\mathcal{X}$, we can associate a rational number $\iota_v$ called *age*. The (even parity) orbifold cohomology group $H^*_{\mathrm{orb}}(\mathcal{X})$ is defined to be

$$H^k_{\mathrm{orb}}(\mathcal{X}) = \bigoplus_{v\in\mathsf{T},\ k-2\iota_v\equiv 0(2)} H^{k-2\iota_v}(\mathcal{X}_v,\mathbb{C}).$$

The degree $k$ of the orbifold cohomology can be a fractional number in general. Each factor $H^*(\mathcal{X}_v,\mathbb{C})$ in the right-hand side is same as the cohomology group of $\mathcal{X}_v$ as a topological space. If not otherwise stated, we will use $\mathbb{C}$ as the coefficient of cohomology groups. We have an involution $\mathrm{inv}\colon I\mathcal{X}\to I\mathcal{X}$ defined by $\mathrm{inv}(x,g)=(x,g^{-1})$. This defines the *orbifold Poincaré pairing*:

$$(\alpha,\beta)_{\mathrm{orb}} := \int_{I\mathcal{X}} \alpha\cup \mathrm{inv}^*(\beta).$$

This pairing is symmetric, non-degenerate over $\mathbb{C}$ and of degree $-2n$, where $n=\dim_{\mathbb{C}}\mathcal{X}$. This involution also induces a map $\mathrm{inv}\colon\mathsf{T}\to\mathsf{T}$. Take a homogeneous $\mathbb{C}$-basis $\{\phi_i\}_{i=1}^N$ of $H^*_{\mathrm{orb}}(\mathcal{X})$. Let $\{\phi^i\}_{i=1}^N$ be the basis dual to $\{\phi_i\}$ with respect to the orbifold Poincaré pairing, *i.e.* $(\phi_i,\phi^j)_{\mathrm{orb}}=\delta_i^j$.

Now assume that the coarse moduli space of $\mathcal{X}$ is projective. The genus zero Gromov-Witten invariants are integrals of the form:

$$\left\langle \alpha_1\psi^{k_1},\dots,\alpha_l\psi^{k_l}\right\rangle^{\mathcal{X}}_{0,l,d} = \int_{[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}} \prod_{i=1}^l \mathrm{ev}_i^*(\alpha_i)\psi_i^{k_i} \tag{19}$$

where $\alpha_i\in H^*_{\mathrm{orb}}(\mathcal{X})$, $d\in H_2(\mathcal{X},\mathbb{Q})$ and $k_i$ is a non-negative integer. The homology class $[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli stack $\mathcal{X}_{0,l,d}$ of genus zero,

$l$-pointed stable maps to $\mathcal{X}$ of degree $d$; $\mathrm{ev}_i\colon \mathcal{X}_{0,l,d} \to I\mathcal{X}$ is the evaluation map[2] at the $i$-th marked point; $\psi_i$ is the first Chern class of the line bundle over $\mathcal{X}_{0,l,d}$ whose fiber at a stable map is the cotangent space of the coarse curve at the $i$-th marked point. We refer the readers to [2] for a more precise definition. (Our notation is taken from [23]; $\mathcal{X}_{0,l,d}$ is denoted by $\mathcal{K}_{0,l}(\mathcal{X},d)$ in [2].) The correlator (19) is non-zero only when $d$ belongs to $\mathrm{Eff}_{\mathcal{X}} \subset H_2(\mathcal{X},\mathbb{Q})$, the semigroup generated by effective stable maps, and $\sum_{i=1}^{l}(\deg \alpha_i + 2k_i) = 2n + 2\langle c_1(T\mathcal{X}), d\rangle + 2l - 6$. The quantum product $\bullet_\tau$ with $\tau \in H^*_{\mathrm{orb}}(\mathcal{X})$ is defined by the formula

$$\phi_i \bullet_\tau \phi_j = \sum_{d\in \mathrm{Eff}_{\mathcal{X}}} \sum_{l\geq 0} \sum_{k=1}^{N} \frac{1}{l!} \langle \phi_i, \phi_j, \tau, \dots, \tau, \phi_k\rangle^{\mathcal{X}}_{0,l+3,d} Q^d \phi^k,$$

where $Q^d$ is an element of the group ring $\mathbb{C}[\mathrm{Eff}_{\mathcal{X}}]$ corresponding to $d \in \mathrm{Eff}_{\mathcal{X}}$. We decompose $\tau = \tau_{0,2} + \tau'$ with $\tau_{0,2} \in H^2(\mathcal{X})$, $\tau' \in \bigoplus_{k\neq 1} H^{2k}(\mathcal{X}) \oplus \bigoplus_{v\in \mathsf{T}'} H^*(\mathcal{X}_v)$. Using the divisor equation [68, 2], we have

$$\phi_i \bullet_\tau \phi_j = \sum_{d\in \mathrm{Eff}_{\mathcal{X}}} \sum_{l\geq 0} \sum_{k=1}^{N} \frac{1}{l!} \langle \phi_i, \phi_j, \tau', \dots, \tau', \phi_k\rangle^{\mathcal{X}}_{0,l+3,d} e^{\langle \tau_{0,2}, d\rangle} Q^d \phi^k. \tag{20}$$

The quantum product is a priori a formal power series in $e^{\tau_{0,2}}Q$ and $\tau'$. When this is a convergent power series, we can put $Q = 1$ and define

$$\circ_\tau := \bullet_\tau|_{Q=1}.$$

Throughout the paper, *we assume that $\circ_\tau$ is convergent on some domain $U \subset H^*_{\mathrm{orb}}(\mathcal{X})$.* The domain $U$ here contains the following limit direction:

$$\Re\langle \tau_{0,2}, d\rangle \to -\infty, \quad \forall d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\}, \quad \tau' \to 0. \tag{21}$$

This is called the *large radius limit*. The (big) orbifold quantum cohomology is a family of associative algebras $(H^*_{\mathrm{orb}}(\mathcal{X}), \circ_\tau)$ parametrized by $\tau \in U$. In the large radius limit, $\circ_\tau$ goes to the orbifold cup product in the sense of Chen-Ruan [19].

3.2. **A-model $\frac{\infty}{2}$VHS.** Let $\{t^i\}_{i=1}^N$ be a co-ordinate system on $H^*_{\mathrm{orb}}(\mathcal{X})$ dual to $\{\phi_i\}$. The orbifold quantum product defines a graded $\frac{\infty}{2}$VHS $\widetilde{\mathcal{F}}$ on $U$

$$\widetilde{\mathcal{F}} := H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}_U\{z\}$$

endowed with a flat *Dubrovin connection* $\nabla$ and a pairing $(\cdot,\cdot)_{\widetilde{\mathcal{F}}}$

$$\nabla := d + \frac{1}{z}\sum_{i=1}^{N} (\phi_i \circ_\tau) dt^i, \quad (f,g)_{\widetilde{\mathcal{F}}} := (f(-z), g(z))_{\mathrm{orb}}, \tag{22}$$

and a grading operator $\mathsf{Gr}$ and Euler vector field $E$

$$\mathsf{Gr} := 2z\partial_z + 2E + 2(\mu + \frac{n}{2}), \quad E := \sum_{i=1}^{N} (1 - \frac{1}{2}\deg \phi_i) t^i \frac{\partial}{\partial t^i} + \sum_{i=1}^{N} r^i \frac{\partial}{\partial t^i},$$

[2]The map $\mathrm{ev}_i$ here is defined only as a map of topological spaces (not as a map of stacks). The evaluation map defined in [2] is a map of stacks but takes values in the *rigidified inertia stack*, which is the same as $I\mathcal{X}$ as a topological space but is different as a stack.

where $n = \dim_{\mathbb{C}} \mathcal{X}$, $c_1(T\mathcal{X}) = \sum_i r^i \phi_i \in H^2(\mathcal{X})$ and $\mu \in \mathrm{End}(H^*_{\mathrm{orb}}(\mathcal{X}))$ is defined by

$$\mu(\phi_i) := \frac{1}{2}(\deg \phi_i - n)\phi_i. \tag{23}$$

The $\frac{\infty}{2}$VHS $\widetilde{\mathcal{F}}$ is also called the *quantum D-module*. The standard argument (as in [27, 51]) and the WDVV equation in orbifold Gromov-Witten theory [2] show that the Dubrovin connection is flat and that the above data satisfy the axioms of a graded $\frac{\infty}{2}$VHS.

Let $H^2(\mathcal{X}, \mathbb{Z})$ denote the cohomology of the constant sheaf $\mathbb{Z}$ on the topological *stack* $\mathcal{X}$ (not on the topological *space*). This group is the set of isomorphism classes of topological orbifold line bundles on $\mathcal{X}$. Let $L_\xi \to \mathcal{X}$ be the orbifold line bundle corresponding to $\xi \in H^2(\mathcal{X}, \mathbb{Z})$. Let $0 \le f_v(\xi) < 1$ be the rational number such that the stabilizer of $\mathcal{X}_v$ ($v \in \mathsf{T}$) acts on $L_\xi|_{\mathcal{X}_v}$ by a complex number $\exp(2\pi\sqrt{-1} f_v(\xi))$. This number $f_v(\xi)$ is called the *age* of $L_\xi$ along $\mathcal{X}_v$. We define a map $G(\xi)\colon H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{X})$ by

$$G(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v) = (\tau_0 - 2\pi\sqrt{-1}\xi_0) \oplus \bigoplus_{v \in \mathsf{T}'} e^{2\pi\sqrt{-1} f_v(\xi)} \tau_v, \tag{24}$$

where $\tau_v \in H^*(\mathcal{X}_v)$ and $\xi_0$ is the image of $\xi$ in $H^2(\mathcal{X}, \mathbb{Q})$. Let $dG(\xi)\colon H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{X})$ be the linear isomorphism given by the differential of $G(\xi)$.

$$dG(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v) = \tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} e^{2\pi\sqrt{-1} f_v(\xi)} \tau_v.$$

**Proposition 3.1.** *The map $\widetilde{\mathcal{F}} \to G(\xi)^* \widetilde{\mathcal{F}}$ given by*

$$\widetilde{\mathcal{F}}_\tau \ni s \longmapsto dG(\xi) s \in \widetilde{\mathcal{F}}_{G(\xi)\tau}$$

*is a homomorphism of graded $\frac{\infty}{2}$VHS's. We call this* Galois action *of $H^2(\mathcal{X}, \mathbb{Z})$ on $\widetilde{\mathcal{F}}$.*

*Proof.* For $\alpha_1, \dots, \alpha_l \in H^*_{\mathrm{orb}}(\mathcal{X})$, we claim that

$$\langle \alpha_1, \dots, \alpha_l \rangle_{0,l,d} = e^{-2\pi\sqrt{-1}\langle \xi_0, d\rangle} \langle dG(\xi)(\alpha_1), \dots, dG(\xi)(\alpha_l) \rangle_{0,l,d}.$$

If there exists an orbifold stable map $f\colon (C, x_1, \dots, x_l) \to \mathcal{X}$ of degree $d$, we have an orbifold line bundle $f^* L_\xi$ on $C$ such that the monodromy at $x_k$ equals $\exp(2\pi\sqrt{-1} f_{v_k}(\xi))$ where $\mathrm{ev}_k(f) \in \mathcal{X}_{v_k}$. Then we must have

$$\deg f^* L_\xi - \sum_{k=1}^{l} f_{v_k} \in \mathbb{Z}, \quad i.e.\ e^{-2\pi\sqrt{-1}\langle \xi_0, d\rangle} \prod_{i=1}^{l} e^{2\pi\sqrt{-1} f_{v_i}(\xi)} = 1.$$

The claim follows from this. The lemma follows from this claim and (20). □

We can assume that $U$ is invariant under the Galois action.

**Definition 3.2.** The *A-model $\frac{\infty}{2}$VHS* of $\mathcal{X}$ is defined to be the quotient $\mathcal{F}$ of $\widetilde{\mathcal{F}} \to U$ by the Galois action of $H^2(\mathcal{X}, \mathbb{Z})$ given above.

$$\mathcal{F} := (\widetilde{\mathcal{F}} \to U)/H^2(\mathcal{X}, \mathbb{Z})$$

The flat connection, the pairing and the grading operator on $\widetilde{\mathcal{F}}$ induce those on $\mathcal{F}$.

3.3. **The space of solutions to quantum differential equations.** As in Section 2.1, the graded $\frac{\infty}{2}$VHS $\widetilde{\mathcal{F}}$ yields a flat connection $\widehat{\nabla}$ on the locally free sheaf $\mathcal{R}^{(0)} = H^*_{\rm orb}(\mathcal{X})\otimes\mathcal{O}_{U\times\mathbb{C}^*}$. This is referred to as the *first structure connection.* A $\widehat{\nabla}$-flat section $s$ of this trivial bundle satisfies the differential equations:

$$\nabla_i s = \widehat{\nabla}_i s = \frac{\partial s}{\partial t^i} + \frac{1}{z}\phi_i \circ_\tau s = 0, \quad i = 1, \dots, N, \tag{25}$$

$$\widehat{\nabla}_{z\partial_z} s = z\frac{\partial s}{\partial z} - \frac{1}{z}E\circ_\tau s + \mu s = 0. \tag{26}$$

We call these equations *quantum differential equations.* We give a fundamental solution $L(\tau, z)$ to the differential equations (25) using gravitational descendants. Let $\mathrm{pr}\colon I\mathcal{X} \to \mathcal{X}$ be the natural projection. We define the action of a class $\tau_0 \in H^*(\mathcal{X})$ on $H^*_{\rm orb}(\mathcal{X})$ by

$$\tau_0 \cdot \alpha = \mathrm{pr}^*(\tau_0) \cup \alpha, \quad \alpha \in H^*_{\rm orb}(\mathcal{X}),$$

where the right-hand side is the cup product on $I\mathcal{X}$. We define

$$L(\tau,z)\phi_i := e^{-\tau_{0,2}/z}\phi_i + \sum_{\substack{(d,l)\neq(0,0)\\ d\in \mathrm{Eff}_{\mathcal{X}}}} \sum_{k=1}^N \frac{\phi^k}{l!}\left\langle \phi_k, \tau', \dots, \tau', \frac{e^{-\tau_{0,2}/z}\phi_i}{-z-\psi}\right\rangle^{\mathcal{X}}_{0,l+2,d} e^{\langle \tau_{0,2}, d\rangle}, \tag{27}$$

where $\tau = \tau_{0,2} + \tau'$ with $\tau_{0,2} \in H^2(\mathcal{X})$ and $\tau' \in \bigoplus_{k\neq 1} H^{2k}(\mathcal{X}) \oplus \bigoplus_{v\in\mathsf{T}'} H^*(\mathcal{X}_v)$ and $1/(-z-\psi)$ in the correlator should be expanded in the series $\sum_{k=0}^\infty (-z)^{-k-1}\psi^k$. It is well-known to specialists that $L(\tau,z)\phi_i$ is a solution to (25).

**Proposition 3.3.** *$L(\tau, z)$ satisfies the following differential equations:*

$$\nabla_k L(\tau,z)\phi_i = 0, \quad k = 1,\dots,N,$$
$$\widehat{\nabla}_{z\partial_z} L(\tau,z)\phi_i = L(\tau,z)(\mu\phi_i - \frac{\rho}{z}\phi_i),$$

*where $\rho := c_1(T\mathcal{X}) \in H^2(\mathcal{X})$. Moreover, $L(\tau,z)$ satisfies*

$$(L(\tau,-z)\phi_i, L(\tau,z)\phi_j)_{\rm orb} = (\phi_i,\phi_j)_{\rm orb},$$
$$dG(\xi)L(G(\xi)^{-1}\tau, z)\alpha = L(\tau,z)e^{-2\pi\sqrt{-1}\xi_0/z}e^{2\pi\sqrt{-1}f_v(\xi)}\alpha, \quad \alpha\in H^*(\mathcal{X}_v),$$

*where $(dG(\xi), G(\xi))$ is the Galois action associated to $\xi \in H^2(\mathcal{X},\mathbb{Z})$. (See Section 3.2.)*

*Proof.* The first equation follows from the topological recursion relation [68, 2.5.5] in orbifold Gromov-Witten theory. The proof for the case of manifolds can be found in [58, Proposition 2], [27, Chapter 10] and the proof for orbifolds is completely parallel. Note that we can decompose $L$ as $L(\tau,z) = S(\tau,z)\circ e^{-\tau_{0,2}/z}$ for some $\mathrm{End}(H^*_{\rm orb}(\mathcal{X}))$-valued function $S(\tau,z)$. By the homogeneity of Gromov-Witten invariants, it is easy to check that $S$ preserves the degree, *i.e.* $\mathsf{Gr}\, S(\tau,z) = S(\tau,z)\,\mathsf{Gr}$. Therefore, $\mathsf{Gr}\, L(\tau,z) = L(\tau,z)(\mathsf{Gr} - 2\rho/z)$. The second equation follows from this and the first equation. Put $\boldsymbol{s}_i' = L(\tau,-z)\phi_i$ and $\boldsymbol{s}_j = L(\tau,z)\phi_j$. By using the first equation and the Frobenius property $(\alpha\circ_\tau\beta,\gamma)_{\rm orb} = (\alpha,\beta\circ_\tau\gamma)_{\rm orb}$, we have

$$\frac{\partial}{\partial t^k}(\boldsymbol{s}_i', \boldsymbol{s}_j)_{\rm orb} = \frac{1}{z}(\phi_k\circ_\tau \boldsymbol{s}_i', \boldsymbol{s}_j)_{\rm orb} - \frac{1}{z}(\boldsymbol{s}_i', \phi_k\circ_\tau \boldsymbol{s}_j)_{\rm orb} = 0.$$

Hence $(\boldsymbol{s}'_i, \boldsymbol{s}_j)_{\mathrm{orb}}$ is constant in $\tau$. When $\tau' = 0$ and $\Re\langle \tau_{0,2}, d\rangle \to -\infty$ for $d \in \mathrm{Eff}_{\mathcal{X}}$, we have asymptotics $\boldsymbol{s}'_i \sim e^{\tau_{0,2}/z}\phi_i$ and $\boldsymbol{s}_j \sim e^{-\tau_{0,2}/z}\phi_j$. Therefore we have

$$(\boldsymbol{s}'_i, \boldsymbol{s}_j)_{\mathrm{orb}} \sim (e^{-\tau_{0,2}/z}\phi_i, e^{\tau_{0,2}/z}\phi_j)_{\mathrm{orb}} = (\phi_i, \phi_j)_{\mathrm{orb}}$$

and the third equation follows. The Galois action sends a $\nabla$-flat section $L(\tau, z)\alpha$ to another $\nabla$-flat section $dG(\xi)L(G(\xi)^{-1}\tau, z)\alpha$. Note that a $\nabla$-flat section $\boldsymbol{s} = L(\tau, z)\phi$ is characterized by the asymptotic initial condition $\boldsymbol{s} \sim e^{-\tau_{0,2}/z}\phi$ in the large radius limit. The fourth equation follows from this and the asymptotics $dG(\xi)L(G(\xi)^{-1}\tau, z)\alpha \sim e^{-\tau_{0,2}/z}e^{-2\pi\sqrt{-1}\xi_0/z}e^{2\pi\sqrt{-1}f_v(\xi)}\alpha$. ◻

Although the convergence of $L(\tau, z)$ is not a priori clear, we know from the differential equations above and the convergence assumption of $\circ_\tau$ that $L(\tau, z)$ is convergent on $(\tau, z) \in U \times \mathbb{C}^*$. Consider the vector space $\mathcal{H}^{\mathcal{X}}$ introduced by Coates-Givental [24]:

$$\mathcal{H}^{\mathcal{X}} := H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}\{z, z^{-1}\}.$$

Recall the space $\mathcal{H}$ of $\nabla$-flat sections introduced in Section 2.2. For the A-model $\frac{\infty}{2}$VHS, $\mathcal{H}$ consists of cohomology-valued functions $s(\tau, z)$ satisfying (25). We identify $\mathcal{H}^{\mathcal{X}}$ with $\mathcal{H}$ by using the fundamental solution $L(\tau, z)$:

$$\mathcal{H}^{\mathcal{X}} \ni \alpha \longmapsto L(\tau, z)\alpha \in \Gamma(U \times \mathbb{C}^*, H^*_{\mathrm{orb}}(\mathcal{X})).$$

Under this identification $\mathcal{H}^{\mathcal{X}} \cong \mathcal{H}$, the embedding $\mathcal{J}_\tau \colon \widetilde{\mathcal{F}}_\tau \to \mathcal{H}^{\mathcal{X}}$ (see (7)) for the A-model $\frac{\infty}{2}$VHS is given by

$$\begin{aligned} \mathcal{J}_\tau(\alpha) &= L(\tau, z)^{-1}\alpha = L(\tau, -z)^\dagger\alpha \\ &= e^{\tau_{0,2}/z}\left(\alpha + \sum_{\substack{(d,l)\neq(0,0)\\ d\in\mathrm{Eff}_{\mathcal{X}}}} \sum_{i=1}^N \frac{1}{l!}\left\langle \alpha, \tau', \dots, \tau', \frac{\phi_i}{z-\psi}\right\rangle_{0,l+2,d}^{\mathcal{X}} e^{\langle \tau_{0,2}, d\rangle}\phi^i\right), \end{aligned} \tag{28}$$

where $L(\tau, -z)^\dagger$ is the adjoint of $L(\tau, -z)$ with respect to $(\cdot, \cdot)_{\mathrm{orb}}$. The second line follows from (27) and an easy computation of the adjoint $L(\tau, -z)^\dagger$. The function $\mathcal{J}_\tau(1)$ is called the *$J$-function*. The image $\mathbb{F}_\tau = \mathcal{J}_\tau(H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}\{z\})$ of the embedding gives the moving subspace realization of the A-model $\frac{\infty}{2}$VHS. By Proposition 3.3, the action of $\widehat{\nabla}_{z\partial_z}$ on $\mathcal{H}^{\mathcal{X}}$ is given by

$$\widehat{\nabla}_{z\partial_z} = z\partial_z + \mu - \frac{\rho}{z}, \tag{29}$$

and the pairing $(\cdot, \cdot)_{\mathcal{H}^{\mathcal{X}}}$ and the symplectic form $\Omega$ on $\mathcal{H}^{\mathcal{X}}$ are given by

$$(\alpha, \beta)_{\mathcal{H}^{\mathcal{X}}} = (\alpha(-z), \beta(z))_{\mathrm{orb}}, \quad \Omega(\alpha, \beta) = \mathrm{Res}_{z=0}\, dz(\alpha, \beta)_{\mathcal{H}^{\mathcal{X}}}. \tag{30}$$

The Galois action on $\widetilde{\mathcal{F}}$ acts on $\nabla$-flat sections as $s(\tau, z) \mapsto dG(\xi)s(G(\xi)^{-1}\tau, z)$. By Proposition 3.3, this induces a map $G^{\mathcal{H}}(\xi) \colon \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^{\mathcal{X}}$ given by

$$G^{\mathcal{H}}(\xi)(\tau_0 \oplus \bigoplus_{v\in\mathsf{T}'} \tau_v) = e^{-2\pi\sqrt{-1}\xi_0/z}\tau_0 \oplus \bigoplus_{v\in\mathsf{T}'} e^{-2\pi\sqrt{-1}\xi_0/z}e^{2\pi\sqrt{-1}f_v(\xi)}\tau_v, \tag{31}$$

where we used the decomposition $\mathcal{H}^{\mathcal{X}} = \bigoplus_{v\in\mathsf{T}} H^*(\mathcal{X}_v) \otimes \mathbb{C}\{z, z^{-1}\}$.

Next we consider the space $\mathcal{V}$ of $\widehat{\nabla}$-flat sections introduced in Section 2.2. For the A-model $\frac{\infty}{2}$VHS, this consists of sections $s$ satisfying both (25) and (26). Then $\mathcal{V}$ is identified with the space of flat sections of the following flat vector bundle $(\mathsf{H}^{\mathcal{X}}, \widehat{\nabla})$:

$$\mathsf{H}^{\mathcal{X}} := H^*_{\mathrm{orb}}(\mathcal{X}) \times \mathbb{C}^* \to \mathbb{C}^*, \quad \widehat{\nabla}_{z\partial_z} = z\partial_z + \mu - \frac{\rho}{z}.$$

Furthermore, we identify the space of flat sections of $(\mathsf{H}^{\mathcal{X}}, \widehat{\nabla}_{z\partial_z})$ with the orbifold cohomology $\mathcal{V}^{\mathcal{X}} := H^*_{\mathrm{orb}}(\mathcal{X})$ via the (well-known) fundamental solution of $\widehat{\nabla}_{z\partial_z} s = 0$:

$$z^{-\mu} z^{\rho} \colon \mathcal{V}^{\mathcal{X}} \to \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{\mathcal{X}}), \quad \phi \mapsto s(z) = e^{-\mu \log z} e^{\rho \log z} \phi. \tag{32}$$

Then we can identify $\mathcal{V}$ with the orbifold cohomology group $\mathcal{V}^{\mathcal{X}} = H^*_{\mathrm{orb}}(\mathcal{X})$. The pairing $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ on $\mathcal{V}^{\mathcal{X}} = H^*_{\mathrm{orb}}(\mathcal{X})$ induced from that on $\mathcal{V}$ (see Equation (6)) is given by

$$(\alpha, \beta)_{\mathcal{V}^{\mathcal{X}}} = (e^{\pi\sqrt{-1}\rho}\alpha, e^{\pi\sqrt{-1}\mu}\beta)_{\mathrm{orb}}. \tag{33}$$

The induced Galois action on $\mathcal{V}^{\mathcal{X}}$ is given by

$$G^{\mathcal{V}}(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v) = e^{-2\pi\sqrt{-1}\xi_0}\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} e^{-2\pi\sqrt{-1}\xi_0} e^{2\pi\sqrt{-1} f_v(\xi)} \tau_v. \tag{34}$$

**Remark 3.4.** The Galois actions on $\mathcal{H}^{\mathcal{X}}$, $\mathcal{V}^{\mathcal{X}}$ are the monodromy transformations of $\nabla$ on $U/H^2(\mathcal{X}, \mathbb{Z})$. The monodromy transformation of $\widehat{\nabla}_{z\partial_z}$ on $\mathbb{C}^*$ is given by

$$e^{-2\pi\sqrt{-1}\mu} e^{2\pi\sqrt{-1}\rho} \colon \mathcal{V}^{\mathcal{X}} \longrightarrow \mathcal{V}^{\mathcal{X}}. \tag{35}$$

This coincides with the Galois action $(-1)^n G^{\mathcal{V}}([K_{\mathcal{X}}])$. Here, $[K_{\mathcal{X}}]$ is the class of the canonical line bundle. When $\mathcal{X}$ is Calabi-Yau, *i.e.* $K_{\mathcal{X}}$ is trivial, the pairing $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ is either symmetric or anti-symmetric depending on whether $n$ is even or odd. In general, this pairing is neither symmetric nor anti-symmetric.

**Proposition 3.5.** *A real (integral) structure on the A-model $\frac{\infty}{2}$VHS $\mathcal{F}$ is given by a real subspace $\mathcal{V}^{\mathcal{X}}_{\mathbb{R}}$ (resp. integral lattice $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$) of $\mathcal{V}^{\mathcal{X}} = H^*_{\mathrm{orb}}(\mathcal{X})$ satisfying*

(i) $\mathcal{V}^{\mathcal{X}} = \mathcal{V}^{\mathcal{X}}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ *(resp.* $\mathcal{V}^{\mathcal{X}} = \mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$*);*
(ii) $\mathcal{V}^{\mathcal{X}}_{\mathbb{R}}$ *(resp.* $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$*) is invariant under the Galois action (34);*
(iii) *The pairing (33) restricted on* $\mathcal{V}^{\mathcal{X}}_{\mathbb{R}}$ *(resp.* $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$*) takes values in* $\mathbb{R}$ *(resp. takes values in* $\mathbb{Z}$ *and is unimodular).*

Let $\kappa_{\mathcal{H}}$ and $\kappa_{\mathcal{V}}$ be the involution of $\mathcal{H}^{\mathcal{X}}$ and $\mathcal{V}^{\mathcal{X}}$ respectively. We discuss basic properties of $\kappa_{\mathcal{H}}$ and $\kappa_{\mathcal{V}}$ below. We decompose the Galois action on $\mathcal{H}^{\mathcal{X}}$ as

$$G^{\mathcal{H}}(\xi) = e^{-2\pi\sqrt{-1}\xi_0/z} G^{\mathcal{H}}_0(\xi), \quad G^{\mathcal{H}}_0(\xi) := \bigoplus_{v \in \mathsf{T}} e^{2\pi\sqrt{-1} f_v(\xi)}.$$

**Proposition 3.6.** *For any real structure on the A-model $\frac{\infty}{2}$VHS, the following holds.*

$$\kappa_{\mathcal{H}}(\tau_{0,2}/z) + (\tau_{0,2}/z)\kappa_{\mathcal{H}} = 0, \quad \kappa_{\mathcal{V}}\tau_{0,2} + \tau_{0,2}\kappa_{\mathcal{V}} = 0, \tag{36}$$

$$G^{\mathcal{H}}_0(\xi)\kappa_{\mathcal{H}} = \kappa_{\mathcal{H}} G^{\mathcal{H}}_0(\xi), \tag{37}$$

$$(z\partial_z + \mu)\kappa_{\mathcal{H}} + \kappa_{\mathcal{H}}(z\partial_z + \mu) = 0, \tag{38}$$

$$\kappa_{\mathcal{H}} = z^{-\mu}\kappa_{\mathcal{V}} z^{\mu}, \quad \textit{when } z \in S^1, \tag{39}$$

*where* $\tau_{0,2} \in H^2(\mathcal{X}, \mathbb{R})$. *Moreover, if* $\mathcal{X}$ *satisfies the following condition:*

$$f_v(\xi) = f_{v'}(\xi),\ \forall \xi \in H^2(\mathcal{X}, \mathbb{Z}) \Longrightarrow v = v', \tag{40}$$

*then we have*

$$\begin{aligned} \kappa_{\mathcal{H}}(H^*(\mathcal{X}_v) \otimes \mathbb{C}\{z, z^{-1}\}) &= H^*(\mathcal{X}_{\mathrm{inv}(v)}) \otimes \mathbb{C}\{z, z^{-1}\}, \\ \kappa_{\mathcal{V}}(H^*(\mathcal{X}_v)) &= H^*(\mathcal{X}_{\mathrm{inv}(v)}). \end{aligned} \tag{41}$$

*When (41) holds,* $\kappa_{\mathcal{V}}$ *satisfies*

$$\kappa_{\mathcal{V}}(\alpha) \in \mathcal{C}(\alpha) + H^{>2k}(\mathcal{X}_{\mathrm{inv}(v)}), \quad \alpha \in H^{2k}(\mathcal{X}_v) \tag{42}$$

*for some complex antilinear isomorphism* $\mathcal{C} : H^{2k}(\mathcal{X}_v) \to H^{2k}(\mathcal{X}_{\mathrm{inv}(v)})$.

*Proof.* Because $G_0^{\mathcal{H}}(\xi)$ is nilpotent and commutes with $e^{-2\pi\sqrt{-1}\xi_0/z}$, $(G^{\mathcal{H}}(\xi))^m = e^{-2\pi\sqrt{-1}m\xi_0/z}$ for $m > 0$ satisfying $f_v(\xi)m \in \mathbb{Z}$ for all $v \in \mathsf{T}$. Hence $\tau_{0,2}/z$ is purely imaginary on $\mathcal{H}^{\mathcal{X}}$ for any $\tau_{0,2} \in H^2(\mathcal{X}, \mathbb{R})$. From (32), we know that the multiplication by $\tau_{0,2}$ is purely imaginary on $\mathcal{V}^{\mathcal{X}}$. Thus we have (36). From $G_0^{\mathcal{H}}(\xi) = e^{2\pi\sqrt{-1}\xi_0/z} G^{\mathcal{H}}(\xi)$, we have (37). Because $\widehat{\nabla}_{z\partial_z} = z\partial_z + \mu - \rho/z$ is purely imaginary on $\mathcal{H}^{\mathcal{X}}$ and so is $\rho/z$, we have (38). By (32), $\kappa_{\mathcal{H}}$ and $\kappa_{\mathcal{V}}$ are related by

$$\kappa_{\mathcal{H}} = z^{-\mu} z^{\rho} \kappa_{\mathcal{V}} z^{-\rho} z^{\mu}, \quad \text{for } z \in S^1.$$

Since $z^\rho = \exp(\rho \log z)$ is real on $\mathcal{V}^{\mathcal{X}}$ when $z \in S^1$, we have (39). Under the condition (40), the decomposition $\mathcal{H}^{\mathcal{X}} = \bigoplus_{v \in \mathsf{T}} H^*(\mathcal{X}_v) \otimes \mathbb{C}\{z, z^{-1}\}$ is the simultaneous eigenspace decomposition for $G_0^{\mathcal{H}}(\xi)$, $\xi \in H_2(\mathcal{X}, \mathbb{Z})$. Therefore, (41) follows from $\overline{e^{2\pi\sqrt{-1}f_v(\xi)}} = e^{2\pi\sqrt{-1}f_{\mathrm{inv}(v)}(\xi)}$ and the reality of $G_0^{\mathcal{H}}(\xi)$. Let $\omega$ be a Kähler class on $\mathcal{X}$. The action of $\omega$ on $H^*(\mathcal{X}_v)$ is nilpotent. In general, a nilpotent operator $\omega$ on a vector space defines an increasing filtration $\{W_k\}_{k \in \mathbb{Z}}$ on it, called a *weight filtration*, which is uniquely determined by the conditions:

$$\omega W_k \subset W_{k-2}, \quad \omega^k \colon \mathrm{Gr}_k^W \cong \mathrm{Gr}_{-k}^W$$

where $\mathrm{Gr}_k^W = W_k / W_{k-1}$. By the Lefschetz decomposition, we know that $W_k = H^{\geq n_v - k}(\mathcal{X}_v)$ in this case ($n_v := \dim_{\mathbb{C}} \mathcal{X}_v$). Since $\kappa_{\mathcal{V}}$ anti-commutes with $\omega$ by (36), $\kappa_{\mathcal{V}}$ preserves this filtration. This shows (42). Here, $\mathcal{C}$ is the isomorphism on the associated graded quotient induced from $\kappa_{\mathcal{V}}$. □

3.4. **Purity and polarization.** For an arbitrary real structure, we study a behavior of the A-model $\frac{\infty}{2}$VHS $\mathcal{F}$ near the large radius limit point (21). We show that it is pure and polarized (in the sense of Definitions 2.8, 2.10) under suitable conditions. Recall that when $\widetilde{\mathcal{F}} \to U$ is pure, this defines a Cecotti-Vafa structure on the vector bundle $K \to U$ by Proposition 2.12.

**Theorem 3.7.** *Let* $\mathcal{X}$ *be a smooth Deligne-Mumford stack with a projective coarse moduli space. Let* $\mathcal{F}$ *be the A-model* $\frac{\infty}{2}$ *VHS of* $\mathcal{X}$ *and take a real structure on* $\mathcal{F}$. *Let* $\omega$ *be a Kähler class on* $\mathcal{X}$.

(i) *If the real structure satisfies (41),* $\mathcal{F}$ *is pure at* $\tau = -x\omega$ *for sufficiently big* $\Re(x) > 0$.

(ii) *If moreover the real structure satisfies (c.f. (42))*

$$
\begin{aligned}
&\kappa_{\mathcal{V}}(\alpha) \in (-1)^k \mathbb{R}_{>0} \operatorname{inv}^*(\overline{\alpha}) + H^{>2k}(\mathcal{X}_{\operatorname{inv}(v)}), \\
\text{or equivalently}\quad &\kappa_{\mathcal{H}}(\alpha) = (-1)^k \mathbb{R}_{>0} \operatorname{inv}^*(\overline{\alpha}) z^{-2k+n_v} + O(z^{-2k+n_v-1})
\end{aligned} \tag{43}
$$

*for* $\alpha \in H^{2k}(\mathcal{X}_v) \subset H^*_{\mathrm{orb}}(\mathcal{X})$, $n_v = \dim_{\mathbb{C}} \mathcal{X}_v$, *then the Hermitian metric* $h(\cdot,\cdot) = g(\kappa(\cdot),\cdot)$ *on the vector bundle* $K \to U$ *satisfies*

$$
(-1)^{\frac{p-q}{2}} h(u,u) > 0, \quad u \in H^{p,q}(\mathcal{X}_v) \subset K_{-x\omega}, \quad u \neq 0
$$

*for sufficiently big* $\Re(x) > 0$, *where we identify* $K_\tau$ *with* $\widetilde{\mathcal{F}}_\tau / z\widetilde{\mathcal{F}}_\tau \cong H^*_{\mathrm{orb}}(\mathcal{X})$. *In particular, if* $H^*_{\mathrm{orb}}(\mathcal{X})$ *consists only of the* $(p,p)$ *part, i.e.* $H^{2p}(\mathcal{X}_v) = H^{p,p}(\mathcal{X}_v)$ *for all* $v \in \mathsf{T}$ *and* $p \geq 0$, *then* $\mathcal{F}$ *is polarized at* $\tau = -x\omega$ *for sufficiently big* $\Re(x) > 0$.

**Remark 3.8.** (i) The condition (41) is satisfied when $\mathcal{X}$ has enough line bundles to separate the inertia components (see (40) in Proposition 3.6). In particular, (41) is always satisfied when $\mathcal{X}$ is a manifold.

(ii) We can consider the *algebraic A-model* $\frac{\infty}{2}$ *VHS*. Let $A^*(\mathcal{X})_{\mathbb{C}}$ denote the Chow ring of $\mathcal{X}$ over $\mathbb{C}$. We set $\mathbb{H}^*(\mathcal{X}_v) := \operatorname{Im}(A^*(\mathcal{X}_v)_{\mathbb{C}} \to H^*(\mathcal{X}_v))$ and define $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X}) := \bigoplus_{v \in \mathsf{T}} \mathbb{H}^*(\mathcal{X}_v)$. The algebraic A-model $\frac{\infty}{2}$VHS is defined to be

$$
\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}_{U \cap \mathbb{H}_{\mathrm{orb}}(\mathcal{X})}\{z\}
$$

with the restriction of Dubrovin connection, the grading operator and pairing, modulo the Galois action given by an element of $\operatorname{Pic}(\mathcal{X})$. Here we used that the quantum product among classes in $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$ again belongs to $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$; this follows from the algebraic construction of orbifold Gromov-Witten theory [2]. When we assume Hodge conjecture for all $\mathcal{X}_v$, each $\mathbb{H}^*(\mathcal{X}_v)$ has Poincaré duality and the orbifold Poincaré pairing is non-degenerate on $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$. In this case, the algebraic A-model $\frac{\infty}{2}$VHS is pure and polarized at $\tau = -x\omega$ for a Kähler class $\omega \in \mathbb{H}^2(\mathcal{X})$ and $\Re(x) \gg 0$ if the conditions corresponding to (41) and (43) are satisfied. The proof below applies to the algebraic A-model $\frac{\infty}{2}$VHS without change. Note that the Poincaré duality of $\mathbb{H}^*(\mathcal{X}_v)$ also implies the Hard Lefschetz of it used in the proof below.

**Remark 3.9.** Hertling [38] and Hertling-Sevenheck [39] studied similar problems for general TERP structures. They considered the change of TERP structures induced by the rescaling $z \mapsto rz$ of the parameter $z$. This rescaling with $r \to \infty$ is called Sabbah orbit in [39] and is equivalent to the flow of minus the Euler vector field: $\tau \mapsto \tau - \rho \log r$ for $\tau \in H^2(\mathcal{X})$. When $\mathcal{X}$ is Fano and $\omega = c_1(\mathcal{X}) = \rho$, the large radius limit corresponds to the Sabbah orbit[3], and the conclusions in Theorem 3.7 can be deduced from a general theorem [39, Theorem 7.3] in this case.

**Remark 3.10.** Singularity theory gives a $\frac{\infty}{2}$VHS with a real structure. According to the recent work of Sabbah [63, Section 4], the $\frac{\infty}{2}$VHS arising from a cohomologically tame function on an affine manifold is pure and polarized. This result covers the case of Landau-Ginzburg model mirror to toric orbifolds treated in Section 4.

[3] The author thanks Claus Hertling for this remark.

The rest of this section is devoted to the proof of Theorem 3.7.

From Equation (28), we see that $e^{x\omega/z}\mathcal{J}_{-x\omega}(\varphi) \to \varphi$ as $\Re(x) \to \infty$. Thus, in the moving subspace realization, the Hodge structure $\mathbb{F}_{-x\omega} = \mathcal{J}_{-x\omega}(\mathcal{F}_{-x\omega})$ has the asymptotics:

$$\mathbb{F}_{-x\omega} \sim e^{-x\omega/z}\mathbb{F}_{\mathrm{lim}} \quad \text{as } \Re(x) \to \infty,$$

where $\mathbb{F}_{\mathrm{lim}} := H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}\{z\}$ is the limiting Hodge structure. This is an analogue of the *nilpotent orbit theorem* [64] in quantum cohomology. First we study the behavior of the nilpotent orbit $x \mapsto e^{-x\omega/z}\mathbb{F}_{\mathrm{lim}}$ for $\Re(x) \gg 0$ (see Proposition 3.13 below).

We study the purity of the $\frac{\infty}{2}$VHS $x \mapsto e^{-x\omega/z}\mathbb{F}_{\mathrm{lim}}$, *i.e.* if the natural map

$$e^{-x\omega/z}\mathbb{F}_{\mathrm{lim}} \cap \kappa_{\mathcal{H}}(e^{-x\omega/z}\mathbb{F}_{\mathrm{lim}}) \longrightarrow e^{-x\omega/z}(\mathbb{F}_{\mathrm{lim}}/z\mathbb{F}_{\mathrm{lim}}) \cong e^{-x\omega/z}H^*_{\mathrm{orb}}(\mathcal{X}) \tag{44}$$

is an isomorphism (see (10) in Proposition 2.9). Under the condition (41), this is equivalent to that the map

$$e^{-x\omega/z}H^*(\mathcal{X}_v)\{z\} \cap \kappa_{\mathcal{H}}(e^{-x\omega/z}H^*(\mathcal{X}_{\mathrm{inv}(v)})\{z\}) \to e^{-x\omega/z}H^*(\mathcal{X}_v)$$

is an isomorphism for each $v \in \mathsf{T}$, where we put $H^*(\mathcal{X}_v)\{z\} = H^*(\mathcal{X}_v) \otimes \mathbb{C}\{z\}$. Since $\kappa_{\mathcal{H}}e^{-x\omega/z} = e^{\overline{x}\omega/z}\kappa_{\mathcal{H}}$ (see (36)), this is equivalent to that

$$H^*(\mathcal{X}_v)\{z\} \cap e^{2t\omega/z}\kappa_{\mathcal{H}}(H^*(\mathcal{X}_{\mathrm{inv}(v)})\{z\}) \to H^*(\mathcal{X}_v), \quad t := \Re(x)$$

is an isomorphism. We further decompose this into $(z\partial_z + \mu)$-eigenspaces. Because $z\partial_z + \mu$ is purely imaginary (38), the above map between the $(z\partial_z + \mu)$-eigenspaces of the eigenvalue $\frac{1}{2}(-k + \mathrm{age}(v) - \mathrm{age}(\mathrm{inv}(v)))$ is of the form:

$$\left(\bigoplus_{l\ge 0} H^{n_v-k-2l}(\mathcal{X}_v)z^l\right) \cap e^{2t\omega/z}\kappa_{\mathcal{H}}\left(\bigoplus_{l\ge 0} H^{n_v+k-2l}(\mathcal{X}_{\mathrm{inv}(v)})z^l\right) \to H^{n_v-k}(\mathcal{X}_v).$$

Here, $n_v = \dim_{\mathbb{C}} \mathcal{X}_v$ and $k$ is an integer such that $n_v - k$ is even. By using (39), we find that this map is conjugate (via $z^{\mu+(k-\iota_v+\iota_{\mathrm{inv}(v)})/2}$) to the following map:

$$H^{\le n_v-k}(\mathcal{X}_v) \cap e^{2t\omega}\kappa_{\mathcal{V}}(H^{\le n_v+k}(\mathcal{X}_{\mathrm{inv}(v)})) \to H^{n_v-k}(\mathcal{X}_v) \tag{45}$$

which is induced by $H^{\le n_v-k}(\mathcal{X}_v) \to H^{\le n_v-k}(\mathcal{X}_v)/H^{\le n_v-k-2}(\mathcal{X}_v) \cong H^{n_v-k}(\mathcal{X}_v)$. We will show that this becomes an isomorphism for $t = \Re(x) \gg 0$ in Lemma 3.12 below.

Let $\mathfrak{a}\colon H^*(\mathcal{X}_v) \to H^{*+2}(\mathcal{X}_v)$ be the operator defined by $\mathfrak{a}(\phi) := \omega \cup \phi$. There exists an operator $\mathfrak{a}^\dagger\colon H^*(\mathcal{X}_v) \to H^{*-2}(\mathcal{X}_v)$ such that $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ generate the Lefschetz $\mathfrak{sl}_2$-action on $H^*(\mathcal{X}_v)$:

$$[\mathfrak{a}, \mathfrak{a}^\dagger] = h, \quad [h, \mathfrak{a}] = 2\mathfrak{a}, \quad [h, \mathfrak{a}^\dagger] = -2\mathfrak{a}^\dagger,$$

where $h := \deg - n_v$ is the (shifted) grading operator. Note that $\mathfrak{a}^\dagger$ is uniquely determined by the above commutation relation and that $\mathfrak{a}^\dagger$ annihilates the primitive cohomology $PH^{n_v-k}(\mathcal{X}_v) := \mathrm{Ker}(\mathfrak{a}^{k+1}\colon H^{n_v-k}(\mathcal{X}_v) \to H^{n_v+k+2}(\mathcal{X}_v))$.

**Lemma 3.11.** *The map $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\colon H^*(\mathcal{X}_v) \to H^*(\mathcal{X}_v)$ sends $H^{\ge n_v-k}(\mathcal{X}_v)$ onto $H^{\le n_v+k}(\mathcal{X}_v)$ isomorphically. Moreover, for $u \in \mathfrak{a}^j PH^{n_v-k-2j}(\mathcal{X}_v) \subset H^{n_v-k}(\mathcal{X}_v)$, one has*

$$e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u = (-1)^{k+j}\frac{j!}{(k+j)!}\omega^k u + H^{<n_v+k}(\mathcal{X}_v).$$

*Proof.* An easy calculation shows that

$$e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\mathfrak{a} = -\mathfrak{a}^\dagger e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}.$$

Therefore, $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}$ should send the weight filtration for the nilpotent operator $\mathfrak{a}$ to that for $\mathfrak{a}^\dagger$. But the weight filtration for $\mathfrak{a}$ is $\{H^{\ge n_v-k}\}_k$ and that for $\mathfrak{a}^\dagger$ is $\{H^{\le n_v+k}\}_k$ (see the proof of Proposition 3.6 for weight filtration). Take $u \in \mathfrak{a}^j PH^{n_v-k-2j}(\mathcal{X}_v)$. Put $u = \mathfrak{a}^j\phi$ for $\phi \in PH^{n_v-k-2j}(\mathcal{X}_v)$. We calculate

$$\begin{aligned} e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u &= e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\mathfrak{a}^j\phi = (-\mathfrak{a}^\dagger)^j e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}\phi = (-\mathfrak{a}^\dagger)^j e^{-\mathfrak{a}}\phi \\ &= (-\mathfrak{a}^\dagger)^j\frac{(-1)^{k+2j}}{(k+2j)!}\mathfrak{a}^{k+2j}\phi + \text{ lower degree term,} \end{aligned}$$

where in the second line we used that $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u \in H^{\le n_v+k}(\mathcal{X}_v)$. Using $\mathfrak{a}^\dagger\mathfrak{a}^l u = l(k+2j+1-l)\mathfrak{a}^{l-1}u$, we arrive at the formula for $e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u$. □

**Lemma 3.12.** *The map (45) is an isomorphism for sufficiently big $t > 0$. Moreover, $u \in H^{n_v-k}(\mathcal{X}_v)$ corresponds to an element of the form*

$$(2t)^{(\deg+k-n_v)/2}(e^{\mathfrak{a}^\dagger}u + O(t^{-1})) \in H^{\le n_v-k}(\mathcal{X}_v) \cap e^{2\omega t}\kappa_{\mathcal{V}}(H^{\le n_v+k}(\mathcal{X}_{\mathrm{inv}(v)}))$$

*under (45), where $(2t)^{\deg/2}$ is defined by $(2t)^{\deg/2} = (2t)^k$ on $H^{2k}(\mathcal{X}_v)$.*

*Proof.* First we rescale (45) by $(2t)^{-\deg/2}$:

$$\begin{array}{ccc} H^{\le n_v-k}(\mathcal{X}_v) \cap e^{2\omega t}\kappa_{\mathcal{V}}(H^{\le n_v+k}(\mathcal{X}_{\mathrm{inv}(v)})) & \longrightarrow & H^{n_v-k}(\mathcal{X}_v) \\ \big\downarrow{\scriptstyle (2t)^{-\deg/2}} & & \big\downarrow{\scriptstyle (2t)^{-\deg/2}} \\ H^{\le n_v-k}(\mathcal{X}_v) \cap e^{\omega}\kappa_t(H^{\le n_v+k}(\mathcal{X}_{\mathrm{inv}(v)})) & \longrightarrow & H^{n_v-k}(\mathcal{X}_v), \end{array}$$

where $\kappa_t := (2t)^{-\deg/2}\kappa_{\mathcal{V}}(2t)^{\deg/2}$. Since the column arrows are isomorphisms for all $t \in \mathbb{R}$, it suffices to show that the bottom arrow is an isomorphism for $t \gg 0$. Observe that the expected dimension of $H^{\le n_v-k}(\mathcal{X}_v) \cap e^{2\omega}\kappa_t(H^{\le n_v+k}(\mathcal{X}_{\mathrm{inv}(v)}))$ equals $\dim H^{n_v-k}(\mathcal{X}_v)$ by Poincaré duality. Thus that the bottom arrow becomes an isomorphism is an open condition for $\kappa_t$. By (42) in Proposition 3.6, we have

$$\kappa_t = \mathcal{C} + O(t^{-1}), \tag{46}$$

for a degree preserving isomorphism $\mathcal{C}\colon H^*(\mathcal{X}_{\mathrm{inv}(v)}) \cong H^*(\mathcal{X}_v)$. Therefore, we only need to check that the map at $t = \infty$

$$H^{\le n_v-k}(\mathcal{X}_v) \cap e^{\mathfrak{a}}H^{\le n_v+k}(\mathcal{X}_v) \to H^{n_v-k}(\mathcal{X}_v) \tag{47}$$

is an isomorphism (recall that $\mathfrak{a} = \omega\cup$). Note that this factors through $\exp(-\mathfrak{a}^\dagger)$ as

$$H^{\le n_v-k} \cap e^{\mathfrak{a}}H^{\le n_v+k} \xrightarrow{\exp(-\mathfrak{a}^\dagger)} H^{\le n_v-k} \cap e^{-\mathfrak{a}^\dagger}e^{\mathfrak{a}}H^{\le n_v+k} \longrightarrow H^{n_v-k},$$

where we omitted the space $\mathcal{X}_v$ from the notation. The second map is induced from the projection $H^{\le n_v-k} \to H^{n_v-k}$ again. Because $e^{-\mathfrak{a}^\dagger}e^{\mathfrak{a}}(H^{\le n_v+k}) = H^{\ge n_v-k}$ by Lemma 3.11, we know that the map (47) is an isomorphism and that the inverse map is given by $u \mapsto \exp(\mathfrak{a}^\dagger)u$. Now the conclusion follows. □

**Proposition 3.13.** *Assume that (41) holds. Then the nilpotent orbit $x \mapsto e^{-x\omega/z}\mathbb{F}_{\rm lim}$ is pure for sufficiently big $t = \Re(x) > 0$ i.e. the map (44) is an isomorphism for $t \gg 0$. The inverse image of $e^{-x\omega/z}u$, $u \in H^{n_v-k}(\mathcal{X}_v)$ under (44) is of the form $e^{-x\omega/z}\varpi_t(u)$ with*

(48)
$$\varpi_t(u) = z^{-\mu-(k-\iota_v+\iota_{\rm inv(v)})/2}(2t)^{(\deg+k-n_v)/2}(e^{\mathfrak{a}^\dagger}u + O(t^{-1})) \in \bigoplus_{l\ge 0} H^{n_v-k-2l}(\mathcal{X}_v)z^l.$$

*When $u = \mathfrak{a}^j\phi$ and $\phi \in PH^{n_v-k-2j}(\mathcal{X}_v)$, we have*

$$(\kappa_{\mathcal{H}}(e^{-x\omega/z}\varpi_t(u)), e^{-x\omega/z}\varpi_t(u))_{\mathcal{H}^{\mathcal{X}}} = \frac{(2t)^k j!}{(k+j)!}\int_{\mathcal{X}_v}\omega^{k+2j}\phi\cup {\rm inv}^*\mathcal{C}(\phi) + O(t^{k-1}) \tag{49}$$

*where $\mathcal{C}\colon H^*(\mathcal{X}_v) \to H^*(\mathcal{X}_{\rm inv(v)})$ is the isomorphism appearing in (42) and $(\cdot,\cdot)_{\mathcal{H}^{\mathcal{X}}}$ is given in (30). If moreover $u \in H^{p,q}(\mathcal{X}_v)\setminus\{0\}$ and the condition (43) holds,*

$$(-1)^{(p-q)/2}(\kappa_{\mathcal{H}}(e^{-x\omega/z}\varpi_t(u)), e^{-x\omega/z}\varpi_t(u))_{\mathcal{H}^{\mathcal{X}}} > 0$$

*for $t = \Re(x) \gg 0$. (Here $p+q = n_v - k$.)*

*Proof.* The purity of $e^{-x\omega/z}\mathbb{F}_{\rm lim}$ and the formula for $\varpi_t(u)$ follow from Lemma 3.12 and the discussion preceding (45). Putting $c = (-k+\iota_v-\iota_{\rm inv(v)})/2$, we calculate

$$\begin{aligned}
&(\kappa_{\mathcal{H}}(e^{-x\omega/z}\varpi_t(u)), e^{-x\omega/z}\varpi_t(u))_{\mathcal{H}^{\mathcal{X}}} = (\kappa_{\mathcal{H}}(\varpi_t(u)), e^{-2t\omega/z}\varpi_t(u))_{\mathcal{H}^{\mathcal{X}}}\\
&= (2t)^{k-n_v}(z^{-\mu-c}\kappa_{\mathcal{V}}(2t)^{\deg/2}(e^{\mathfrak{a}^\dagger}u+O(t^{-1})), z^{-\mu+c}(2t)^{\deg/2}(e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u+O(t^{-1})))_{\mathcal{H}^{\mathcal{X}}}\\
&= (2t)^k((-1)^{-\mu-c}\kappa_t(e^{\mathfrak{a}^\dagger}u+O(t^{-1})), e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u+O(t^{-1}))_{\rm orb}
\end{aligned}$$

where we used $e^{-2t\omega/z}z^{-\mu}(2t)^{\deg/2} = z^{-\mu}(2t)^{\deg/2}e^{-\mathfrak{a}}$ and (39) in the second line (we assume $|z|=1$) and set $\kappa_t := (2t)^{-\deg/2}\kappa_{\mathcal{V}}(2t)^{\deg/2}$ again in the third line. From (46), the highest order term in $t$ becomes

$$(2t)^k((-1)^{-\mu-c}e^{-\mathfrak{a}^\dagger}\mathcal{C}(u), e^{-\mathfrak{a}}e^{\mathfrak{a}^\dagger}u)_{\rm orb}.$$

Note that $\mathcal{C}$ anticommutes with $\mathfrak{a}, \mathfrak{a}^\dagger$ by (36). By a calculation using Lemma 3.11, we find that this equals the highest order term of the right-hand side of (49). The last statement on positivity follows from the classical Hodge-Riemann bilinear inequality:

$$(-1)^{(p-q)/2}(-1)^{(n_v-k)/2-j}\int_{\mathcal{X}_v}\omega^{k+2j}\phi\cup\overline{\phi} > 0$$

for $\phi \in PH^{n_v-k-2j}(\mathcal{X}_v)\cap H^{p-j,q-j}(\mathcal{X}_v)\setminus\{0\}$, $n_v-k$ even. ☐

Next we show that $x \mapsto \mathbb{F}_{-x\omega}$ is pure for $t = \Re(x) \gg 0$. We set $\mathbb{F}'_{-x\omega} = e^{x\omega/z}\mathbb{F}_{-x\omega}$. Again by (10) in Proposition 2.9 and $\kappa_{\mathcal{H}}e^{-x\omega/z} = e^{\overline{x}\omega/z}\kappa_{\mathcal{H}}$, it is sufficient to show that

$$\mathbb{F}'_{-x\omega}\cap e^{2t\omega/z}\kappa_{\mathcal{H}}(\mathbb{F}'_{-x\omega}) \longrightarrow \mathbb{F}'_{-x\omega}/z\mathbb{F}'_{-x\omega}$$

is an isomorphism. Put $\kappa^t = e^{2t\omega/z}\kappa_{\mathcal{H}}$ ($\kappa^t$ is different from $\kappa_t$ appearing in (46)). Fix a basis $\{\phi_1,\dots,\phi_N\}$ of $H^*_{\rm orb}(\mathcal{X})$. Define an $N\times N$ matrix $A_t(z,z^{-1})$ by

$$[\kappa^t(\phi_1),\dots,\kappa^t(\phi_N)] = [\phi_1,\dots,\phi_N]A_t(z,z^{-1}). \tag{50}$$

This matrix $A_t$ is a Laurent polynomial in $z$ (by (39)) and a polynomial in $t$. We already showed that (44) is an isomorphism for $t = \Re(x) \gg 0$. Therefore, $A_t(z)$ admits the

Birkhoff factorization $A_t(z) = B_t(z)C_t(z)$ for $t \gg 0$, where $B_t \colon \mathbb{D}_0 \to GL_N(\mathbb{C})$ with $B_t(0) = \mathbf{1}$ and $C_t \colon \mathbb{D}_\infty \to GL_N(\mathbb{C})$ (see Remark 2.11). The matrix $B_t(z)$ here is given by

$$[\varpi_t(\phi_1), \dots, \varpi_t(\phi_N)] = [\phi_1, \dots, \phi_N]B_t(z)$$

for $\varpi_t(\phi_i)$ appearing in (48). In particular, $B_t(z)$ and $C_t(z)$ are polynomials in $z$ and $z^{-1}$ respectively and have at most polynomial growth in $t$. We define $Q_x \colon \mathbb{P}^1 \setminus \{0\} \to GL_N(\mathbb{C})$ by

$$[j_1, \dots, j_N] = [\phi_1, \dots, \phi_N]Q_x(z), \quad j_i := e^{x\omega/z}\mathcal{J}_{-x\omega}(\phi_i) \tag{51}$$

where $\mathcal{J}_\tau$ is given in (28). The vectors $j_1, \dots, j_N$ form a basis of $\mathbb{F}'_{-t\omega}$ and $Q_x(\infty) = \mathbf{1}$. Note that $Q_x = \mathbf{1} + O(e^{-\epsilon_0 t})$ as $t = \Re(x) \to \infty$ for $\epsilon_0 := \min(\langle \omega, d\rangle \ ; \ d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\})$. From (50) and (51), we find

$$[\kappa^t(j_1), \dots, \kappa^t(j_N)] = [j_1, \dots, j_N]Q_x^{-1}A_t\overline{Q}_x,$$

where $\overline{Q}_x$ is the complex conjugate of $Q_x$ with $z$ restricted to $S^1 = \{|z| = 1\}$. As we did in Remark 2.11, it suffices to show that $Q_x^{-1}A_t\overline{Q}_x$ admits the Birkhoff factorization. We have

$$Q_x^{-1}A_t\overline{Q}_x = B_t(B_t^{-1}Q_x^{-1}B_t)(C_t\overline{Q}_xC_t^{-1})C_t$$

and for $0 < \epsilon < \epsilon_0$,

$$B_t^{-1}Q_x^{-1}B_t = \mathbf{1} + O(e^{-\epsilon t}), \quad C_t\overline{Q}_xC_t^{-1} = \mathbf{1} + O(e^{-\epsilon t}), \quad \text{as } t = \Re(x) \to \infty.$$

Here we used that $B_t$ and $C_t$ have at most polynomial growth in $t$. By the continuity of Birkhoff factorization, $(B_t^{-1}Q_x^{-1}B_t)(C_t\overline{Q}_xC_t^{-1}) = \mathbf{1} + O(e^{-\epsilon t})$ admits the Birkhoff factorization of the form:

$$(B_t^{-1}Q_x^{-1}B_t)(C_t\overline{Q}_xC_t^{-1}) = \tilde{B}_x(z)\tilde{C}_x(z), \quad \tilde{B}_x = \mathbf{1} + O(e^{-\epsilon t}), \quad \tilde{C}_x = \mathbf{1} + O(e^{-\epsilon t}) \tag{52}$$

for $t = \Re(x) \gg 0$, where $\tilde{B}_x \colon \mathbb{D}_0 \to GL_N(\mathbb{C})$, $\tilde{B}_x(0) = \mathbf{1}$ and $\tilde{C}_x \colon \mathbb{D}_\infty \to GL_N(\mathbb{C})$. The order estimate $O(e^{-\epsilon t})$ holds in the $C^0$-norm on the loop space $C^\infty(S^1, \mathrm{End}(\mathbb{C}^N))$. See Appendix 7.1 for the proof of the order estimate in (52). Therefore $Q_x^{-1}A_t\overline{Q}_x$ also has the Birkhoff factorization for $t = \Re(x) \gg 0$ and we know that

$$[\Pi_x(\phi_1), \dots, \Pi_x(\phi_N)] := [\phi_1, \dots, \phi_N]Q_x(z)B_t(z)\tilde{B}_x(z)$$

form a basis of $\mathbb{F}'_{-x\omega} \cap \kappa^t(\mathbb{F}'_{-x\omega})$, *i.e.* $e^{-x\omega/z}\Pi_x(\phi_1), \dots, e^{-x\omega/z}\Pi_x(\phi_N)$ form a basis of $\mathbb{F}_{-x\omega} \cap \kappa_{\mathcal{H}}(\mathbb{F}_{-x\omega})$. Using that $\Pi_x(\phi_i) = \varpi_x(\phi_i) + O(e^{-\epsilon t})$ and Proposition 3.13, we have

$$(-1)^{(p-q)/2}(\kappa_{\mathcal{H}}(e^{-x\omega/z}\Pi_x(\phi)), e^{-x\omega/z}\Pi_x(\phi))_{\mathcal{H}^{\mathcal{X}}} > 0, \quad \phi \in H^{p,q}(\mathcal{X}_v) \setminus \{0\}$$

for sufficiently big $\Re(x) > 0$. This completes the proof of Theorem 3.7.

3.5. **An A-model integral structure.** Real or integral structures in the A-model $\frac{\infty}{2}$VHS (in the sense of Definition 2.2) are not unique. In this section, we construct an example of A-model integral structures which comes from $K$-theory and will make sense for general symplectic orbifolds. By Theorem 3.7, this yields a Cecotti-Vafa structure near the large radius limit point. For weak Fano toric orbifolds, assuming Conjecture 4.15, we will show in Section 4 that this A-model integral structure coincides with the singularity B-model integral structure under mirror symmetry.

Let $K(\mathcal{X})$ denote the Grothendieck group of topological orbifold vector bundles on $\mathcal{X}$. See *e.g.* [3, 53] for vector bundles on orbifolds. For an orbifold vector bundle $\widetilde{V}$ on the inertia stack $I\mathcal{X}$, we have an eigenbundle decomposition of $\widetilde{V}|_{\mathcal{X}_v}$

$$\widetilde{V}|_{\mathcal{X}_v} = \bigoplus_{0\le f<1} \widetilde{V}_{v,f}$$

with respect to the action of the stabilizer of $\mathcal{X}_v$. Here, the stabilizer acts on $\widetilde{V}_{v,f}$ by $\exp(2\pi\sqrt{-1}f) \in \mathbb{C}$. Let $\mathrm{pr}\colon I\mathcal{X} \to \mathcal{X}$ be the projection. The Chern character map $\widetilde{\mathrm{ch}}\colon K(\mathcal{X}) \to H^*(I\mathcal{X})$ is defined by

$$\widetilde{\mathrm{ch}}(V) := \bigoplus_{v\in\mathsf{T}} \sum_{0\le f<1} e^{2\pi\sqrt{-1}f}\,\mathrm{ch}((\mathrm{pr}^* V)_{v,f})$$

where ch is the ordinary Chern character and $V$ is an orbifold vector bundle on $\mathcal{X}$. For an orbifold vector bundle $V$ on $\mathcal{X}$, let $\delta_{v,f,i}$, $i = 1,\dots,l_{v,f}$ be the Chern roots of $(\mathrm{pr}^* V)_{v,f}$. The Todd class $\widetilde{\mathrm{Td}}\colon K(\mathcal{X}) \to H^*(I\mathcal{X})$ is defined by

$$\widetilde{\mathrm{Td}}(V) = \bigoplus_{v\in\mathsf{T}} \prod_{0<f<1, 1\le i\le l_{v,f}} \frac{1}{1-e^{-2\pi\sqrt{-1}f}e^{-\delta_{v,f,i}}} \prod_{f=0, 1\le i\le l_{v,0}} \frac{\delta_{v,0,i}}{1-e^{-\delta_{v,0,i}}}$$

We put $\mathrm{Td}_{\mathcal{X}} := \widetilde{\mathrm{Td}}(T\mathcal{X})$. These characteristic classes appear in the following theorem.

**Theorem 3.14** (Orbifold Riemann-Roch [48, 67])**.** *Let $V$ be a holomorphic orbifold vector bundle on $\mathcal{X}$. The holomorphic Euler characteristic $\chi(V) := \sum_{i=0}^{\dim\mathcal{X}}(-1)^i \dim H^i(\mathcal{X}, V)$ is given by*

$$\chi(V) = \int_{I\mathcal{X}} \widetilde{\mathrm{ch}}(V) \cup \mathrm{Td}_{\mathcal{X}}\,. \tag{53}$$

Introduce another multiplicative characteristic class $\widehat{\Gamma}\colon K(\mathcal{X}) \to H^*(I\mathcal{X})$ as

$$\widehat{\Gamma}(V) := \bigoplus_{v\in\mathsf{T}} \prod_{0\le f<1} \prod_{i=1}^{l_{v,f}} \Gamma(1-f+\delta_{v,f,i}) \in H^*(I\mathcal{X}),$$

where $\delta_{v,f,i}$ is the same as above. The Gamma function on the right-hand side should be expanded in series at $1-f>0$. We put $\widehat{\Gamma}_{\mathcal{X}} := \widehat{\Gamma}(T\mathcal{X})$. When defining the A-model integral structure below, we need to assume the following:

- (A1) The map $\widetilde{\mathrm{ch}}\colon K(\mathcal{X}) \to H^*(I\mathcal{X})$ becomes an isomorphism after tensored with $\mathbb{C}$.
- (A2) The right-hand side of the orbifold Riemann-Roch formula (53) takes values in $\mathbb{Z}$ for any complex orbifold vector bundle $V$. Define $\chi(V)$ to be the value of the right-hand side of (53) for any orbifold vector bundle $V$.

(A3) The pairing $(V_1, V_2) \mapsto \chi(V_1 \otimes V_2)$ on $K(\mathcal{X})$ induces a surjective map $K(\mathcal{X}) \to \mathrm{Hom}(K(\mathcal{X}), \mathbb{Z})$.

**Remark 3.15.** (i) When $\mathcal{X}$ is a quotient orbifold of the form $\mathcal{X} = Y/G$, where $Y$ is a compact manifold and $G$ is a compact Lie group acting on $Y$ with at most finite stabilizers, (A1) follows from Adem-Ruan's decomposition theorem [3, Theorem 5.1]. Note that an orbifold without generic stabilizers can be presented as a quotient orbifold $Y/G$ (see *e.g.* [3]).

(ii) When $\mathcal{X}$ is again a quotient orbifold $Y/G$, (A2) follows from Kawasaki's index theorem [49] for elliptic operators on orbifolds (whose proof uses the $G$-equivariant index). The right-hand side of (53) becomes the index of a certain elliptic operator $\overline{\partial}+\overline{\partial}^*\colon V\otimes\Omega^{0,\mathrm{even}}_{\mathcal{X}} \to V\otimes\Omega^{0,\mathrm{odd}}_{\mathcal{X}}$, where $\overline{\partial}$ is a not necessarily integrable $(0,1)$ connection and $\overline{\partial}^*$ is its adjoint. The author does not know a purely topological proof.

(iii) (A3) would follow from a universal coefficient theorem and Poincaré duality for orbifold $K$-theory (which are true for manifolds), but the author does not know a proof nor a reference.

**Definition-Proposition 3.16.** *Define $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \subset \mathcal{V}^{\mathcal{X}} = H^*_{\mathrm{orb}}(\mathcal{X})$ as the image of the map*

$$\Psi\colon K(\mathcal{X}) \longrightarrow \mathcal{V}^{\mathcal{X}}, \quad [V] \longmapsto \frac{1}{(2\pi)^{n/2}} \widehat{\Gamma}_{\mathcal{X}} \cup (2\pi\sqrt{-1})^{\deg/2} \operatorname{inv}^*(\widetilde{\mathrm{ch}}(V)), \tag{54}$$

*where* $\deg\colon H^*(I\mathcal{X}) \to H^*(I\mathcal{X})$ *is a grading operator on $H^*(I\mathcal{X})$ defined by* $\deg = 2k$ *on $H^{2k}(I\mathcal{X})$ and $\cup$ is the cup product in $H^*(I\mathcal{X})$. Then*

(i) *Under the assumption (A1) above, $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$ is a lattice in $\mathcal{V}^{\mathcal{X}}$ such that $\mathcal{V}^{\mathcal{X}} \cong \mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$.*

(ii) *The Galois action $G^{\mathcal{V}}(\xi)$ on $\mathcal{V}^{\mathcal{X}}$ in (34) corresponds to tensoring by the line bundle $\otimes L_\xi^\vee$ in $K(\mathcal{X})$, i.e. $\Psi([V \otimes L_\xi^\vee]) = G^{\mathcal{V}}(\xi)(\Psi([V]))$.*

(iii) *The pairing $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ on $\mathcal{V}^{\mathcal{X}}$ in (33) corresponds to the Mukai pairing on $K(\mathcal{X})$ defined by $([V_1],[V_2])_{K(\mathcal{X})} := \chi(V_2^\vee \otimes V_1)$, i.e. $(\Psi([V_1]), \Psi([V_2]))_{\mathcal{V}^{\mathcal{X}}} = ([V_1],[V_2])_{K(\mathcal{X})}$. In particular, the pairing $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ restricted on $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$ takes values in $\mathbb{Z}$ under assumption (A2) and is unimodular under assumption (A3).*

*Therefore $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$ satisfies the conditions in Proposition 3.5 and defines an integral structure in the A-model. We call $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$ a* $\widehat{\Gamma}$-integral structure. *The real involution $\kappa_{\mathcal{V}}$ on $\mathcal{V}^{\mathcal{X}}$ for the $\widehat{\Gamma}$-integral structure is given by*

$$\kappa_{\mathcal{V}}(\alpha) = (-1)^k \prod_{0\le f<1} \prod_{i=1}^{l_{\mathrm{inv}(v),f}} \frac{\Gamma(1-f+\delta_{\mathrm{inv}(v),f,i})}{\Gamma(1-\overline{f}-\delta_{\mathrm{inv}(v),f,i})} \operatorname{inv}^* \overline{\alpha}, \quad \alpha \in H^{2k}(\mathcal{X}_v) \subset \mathcal{V}^{\mathcal{X}},$$

*where $\delta_{\mathrm{inv}(v),f,i}$, $i = 1, \dots, l_{\mathrm{inv}(v),f}$ are the Chern roots of $(\mathrm{pr}^* T\mathcal{X})_{\mathrm{inv}(v),f}$ and*

$$\overline{f} := \begin{cases} 1-f & \text{if } 0<f<1 \\ 0 & \text{if } f=0. \end{cases}$$

*Therefore, this $\kappa_{\mathcal{V}}$ satisfies (41) and (43). In particular, the conclusions of Theorem 3.7 hold for the $\widehat{\Gamma}$-integral structure on the A-model $\frac{\infty}{2}$ VHS.*

*Proof.* Because $\widehat{\Gamma}_{\mathcal{X}}\cup$ and $(2\pi\sqrt{-1})^{\deg/2}$ are invertible operators over $\mathbb{C}$, (A1) implies (i). It is easy to check the second statement (ii). For (iii), we calculate

$$
\begin{aligned}
(\Psi(V_1),\Psi(V_2))_{\mathcal{V}^{\mathcal{X}}} &= (e^{\pi\sqrt{-1}\mu}\Psi(V_2), e^{\pi\sqrt{-1}\rho}\Psi(V_1))_{\mathrm{orb}} \\
&= \frac{1}{(2\pi)^n}\sum_{v\in\mathsf{T}}(2\pi\sqrt{-1})^{\dim\mathcal{X}_v}\times
\end{aligned}
$$

$$
\int_{\mathcal{X}_v}\prod_{f,i}\Gamma(1-f-\tfrac{\delta_{v,f,i}}{2\pi\sqrt{-1}})\Gamma(1-\overline{f}+\tfrac{\delta_{v,f,i}}{2\pi\sqrt{-1}})\cdot e^{\frac{\rho}{2}}\widetilde{\mathrm{ch}}(V_1)_v\cdot e^{\pi\sqrt{-1}(\iota_v-\frac{n}{2}+\frac{\deg}{2})}\widetilde{\mathrm{ch}}(V_2)_{\mathrm{inv}(v)},
$$

where $\widetilde{\mathrm{ch}}(V_1)_v$ and $\widetilde{\mathrm{ch}}(V_2)_{\mathrm{inv}(v)}$ are $H^*(\mathcal{X}_v)$ and $H^*(\mathcal{X}_{\mathrm{inv}(v)})$ components of $\widetilde{\mathrm{ch}}(V_1)$ and $\widetilde{\mathrm{ch}}(V_2)$ respectively. Using $\Gamma(1-z)\Gamma(z)=\pi/\sin(\pi z)$ and $\sum_{f,i}\delta_{v,f,i}=\mathrm{pr}^*\rho|_{\mathcal{X}_v}$, we calculate

$$
\prod_{f,i}\Gamma(1-f-\tfrac{\delta_{v,f,i}}{2\pi\sqrt{-1}})\Gamma(1-\overline{f}+\tfrac{\delta_{v,f,i}}{2\pi\sqrt{-1}}) = (2\pi\sqrt{-1})^{n-\dim\mathcal{X}_v}e^{-\frac{\rho}{2}}e^{-\pi\sqrt{-1}\iota_v}\,\mathrm{Td}_{\mathcal{X},v}\,.
$$

Putting these together, we have $(\Psi(V_1),\Psi(V_2))_{\mathcal{V}^{\mathcal{X}}}=\chi(V_2^\vee\otimes V_1)$. A straightforward calculation shows the rest of the statements. □

**Remark 3.17.** Instead of working with topological $K$-groups, we can use the $K$-group of algebraic vector bundles (or coherent sheaves) on the smooth Deligne-Mumford stack $\mathcal{X}$. In this case, the map $\Psi$ in Definition-Proposition 3.16 defines an integral structure on the algebraic A-model $\frac{\infty}{2}$VHS introduced in Remark 3.8.

## 4. Integral structures via toric mirrors

In previous sections, we studied properties of arbitrary real and integral structures. In this section, we use mirror symmetry to find the "most natural" integral structure in the A-model. We calculate the integral structures in the singularity mirrors (Landau-Ginzburg model) of toric orbifolds and study the pulled back integral structures in orbifold quantum cohomology.

4.1. **Toric orbifolds.** To fix the notation, we give the definition of toric orbifolds and collect several facts. By a toric orbifold, we mean a toric Deligne-Mumford stack in the sense of Borisov-Chen-Smith [9]. We only deal with a compact toric orbifold with a projective coarse moduli space and define a toric orbifold as a quotient of $\mathbb{C}^m$ by a *connected* torus $\mathbb{T}\cong(\mathbb{C}^*)^r$. For toric orbifolds/varieties, we refer the reader to [9, 56, 33, 6].

4.1.1. *Definition.* We begin with the following data:

- an $r$-dimensional algebraic torus $\mathbb{T}\cong(\mathbb{C}^*)^r$; we set $\mathbb{L}:=\mathrm{Hom}(\mathbb{C}^*,\mathbb{T})$;
- $m$ elements $D_1,\dots,D_m\in\mathbb{L}^\vee=\mathrm{Hom}(\mathbb{T},\mathbb{C}^*)$ such that $\mathbb{L}^\vee\otimes\mathbb{R}=\sum_{i=1}^m\mathbb{R}D_i$;
- a vector $\eta\in\mathbb{L}^\vee\otimes\mathbb{R}$.

The elements $D_1,\dots,D_m$ define a homomorphism $\mathbb{T}\to(\mathbb{C}^*)^m$. Let $\mathbb{T}$ act on $\mathbb{C}^m$ via this homomorphism. The vector $\eta$ defines a stability condition of this torus action. Set

$$
\mathcal{A}:=\{I\subset\{1,\dots,m\}\ ;\ \sum_{i\in I}\mathbb{R}_{>0}D_i\ni\eta\}.
$$

We define a quotient stack $\mathcal{X}$ to be

$$\mathcal{X} = [\mathcal{U}_\eta/\mathbb{T}], \quad \mathcal{U}_\eta := \mathbb{C}^m \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I,$$

where $\mathbb{C}^I := \{(z_1, \dots, z_m) \; ; \; z_i = 0 \text{ for } i \notin I\}$. Under the following conditions, $\mathcal{X}$ is a smooth Deligne-Mumford stack with a projective coarse moduli space:

(A) $\{1, \dots, m\} \in \mathcal{A}$.
(B) $\sum_{i \in I} \mathbb{R} D_i = \mathbb{L}^\vee \otimes \mathbb{R}$ for $I \in \mathcal{A}$.
(C) $\{(c_1, \dots, c_m) \in \mathbb{R}^m_{\ge 0} \; ; \; \sum_{i=1}^m c_i D_i = 0\} = \{0\}$.

The conditions (A), (B) and (C) ensure that $\mathcal{X}$ is non-empty, that the stabilizer is finite and that $\mathcal{X}$ is compact respectively. The generic stabilizer of $\mathcal{X}$ is given by the kernel of $\mathbb{T} \to (\mathbb{C}^*)^m$ and $\dim_\mathbb{C} \mathcal{X} = n := m - r$.

We can also construct $\mathcal{X}$ as a symplectic quotient as follows (see also [6]). Let $\mathbb{T}_\mathbb{R}$ denote the maximal compact subgroup of $\mathbb{T}$ isomorphic to $(S^1)^r$. Let $\mathfrak{h} \colon \mathbb{C}^m \to \mathbb{L}^\vee \otimes \mathbb{R}$ be the moment map for the $\mathbb{T}_\mathbb{R}$-action on $\mathbb{C}^m$:

$$\mathfrak{h}(z_1, \dots, z_m) = \sum_{i=1}^m |z_i|^2 D_i.$$

The $\mathbb{T}_\mathbb{R}$-action on the level set $\mathfrak{h}^{-1}(\eta)$ has only finite stabilizers and we have an isomorphism of symplectic orbifolds:

$$\mathcal{X} \cong \mathfrak{h}^{-1}(\eta)/\mathbb{T}_\mathbb{R}. \tag{55}$$

By renumbering the indices if necessary, we can assume that

$$\{1, \dots, m\} \setminus \{i\} \in \mathcal{A} \quad \text{if and only if} \quad 1 \le i \le m'$$

where $m'$ is less than or equal to $m$. We can easily check that $I \supset \{m'+1, \dots, m\}$ for any $I \in \mathcal{A}$ and $D_{m'+1}, \dots, D_m$ are linearly independent over $\mathbb{R}$. The elements $D_1, \dots, D_m$ define the following exact sequence

$$0 \longrightarrow \mathbb{L} \xrightarrow{(D_1, \dots, D_m)} \mathbb{Z}^m \xrightarrow{\ \beta\ } \boldsymbol{N} \longrightarrow 0, \tag{56}$$

where $\boldsymbol{N}$ is a finitely generated abelian group. By the long exact sequence associated with the functor $\mathrm{Tor}_\bullet(-, \mathbb{C}^*)$, we find that the torsion part $\boldsymbol{N}_{\mathrm{tor}} = \mathrm{Tor}_1(\boldsymbol{N}, \mathbb{C}^*)$ of $\boldsymbol{N}$ is isomorphic to the generic stabilizer $\mathrm{Ker}(\mathbb{T} \to (\mathbb{C}^*)^m)$. The free part $\boldsymbol{N}_{\mathrm{free}} = \boldsymbol{N}/\boldsymbol{N}_{\mathrm{tor}}$ is of rank $n = \dim_\mathbb{C} \mathcal{X}$. Let $b_1, \dots, b_m$ be the images in $\boldsymbol{N}$ of the standard basis of $\mathbb{Z}^m$ under $\beta$. The *stacky fan* of $\mathcal{X}$, in the sense of Borisov-Chen-Smith [9], is given by the following data:

- vectors $b_1, \dots, b_{m'}$ in $\boldsymbol{N}$;
- a complete simplicial fan $\Sigma$ in $\boldsymbol{N} \otimes \mathbb{R}$ such that
  - (i) the set of one dimensional cones is $\{\mathbb{R}_{\ge 0} b_1, \dots, \mathbb{R}_{\ge 0} b_m\}$;
  - (ii) $\sigma_I = \sum_{i \notin I} \mathbb{R}_{\ge 0} b_i$ defines a cone of $\Sigma$ if and only if $I \in \mathcal{A}$.

The toric variety defined by the fan $\Sigma$ is the coarse moduli space of $\mathcal{X}$. The conditions (B) and (C) correspond to that $\Sigma$ is simplicial and that $\Sigma$ is complete, *i.e.* the union of all cones in $\Sigma$ is $\boldsymbol{N} \otimes \mathbb{R}$. We may refer to $\mathcal{A}$ as the set of "anticones"[4].

[4]This name is due to Tom Coates.

**Remark 4.1.** Borisov-Chen-Smith [9] defined a toric Deligne-Mumford stack starting from data of a stacky fan. Our construction can give every toric Deligne-Mumford stack in their sense which has a projective coarse moduli space. Note that the vectors $b_{m'+1},\dots,b_m$ do not appear as data of a stacky fan. These redundant information in our initial data makes $\beta$ surjective and allows us to define $\mathcal{X}$ as a quotient by a *connected* torus $\mathbb{T}$.

4.1.2. *Kähler cone and a choice of a nef basis.* Since every element of $\mathcal{A}$ contains $\{m'+1,\dots,m\}$, it is convenient to put

$$\mathcal{A}' = \{I' \subset \{1,\dots,m'\} \ ; \ I' \cup \{m'+1,\dots,m\} \in \mathcal{A}\}.$$

We can easily see that $\mathcal{U}_\eta$ factors as

$$\mathcal{U}_\eta = \mathcal{U}'_\eta \times (\mathbb{C}^*)^{m-m'}, \quad \mathcal{U}'_\eta = \mathbb{C}^{m'} \setminus \bigcup_{I' \notin \mathcal{A}'} \mathbb{C}^{I'}.$$

Thus we can write

$$\mathcal{X} = [\mathcal{U}'_\eta/\mathbb{G}], \quad \mathbb{G} := \operatorname{Ker}(\mathbb{T} \to (\mathbb{C}^*)^m \to (\mathbb{C}^*)^{\{m'+1,\dots,m\}}).$$

Note that $\mathbb{G}$ is isomorphic to $(\mathbb{C}^*)^{r'}$ times a finite abelian group for $r' := r-(m-m')$. Every character $\xi\colon \mathbb{G} \to \mathbb{C}^*$ of $\mathbb{G}$ defines an orbifold line bundle $L_\xi := \mathbb{C} \times_{\xi,\mathbb{G}} \mathcal{U}'_\eta \to \mathcal{X}$. Under this correspondence between $\xi$ and $L_\xi$, the Picard group $\operatorname{Pic}(\mathcal{X})$ is identified with the character group $\operatorname{Hom}(\mathbb{G},\mathbb{C}^*)$ and also with $H^2(\mathcal{X},\mathbb{Z})$ (via $c_1$):

$$\operatorname{Pic}(\mathcal{X}) \cong \operatorname{Hom}(\mathbb{G},\mathbb{C}^*) \cong \mathbb{L}^\vee / \textstyle\sum_{i=m'+1}^m \mathbb{Z}D_i \cong H^2(\mathcal{X},\mathbb{Z}).$$

The image $\overline{D}_i$ of $D_i$ in $H^2(\mathcal{X},\mathbb{R})$ is the Poincaré dual of the toric divisor $\{z_i=0\} \subset \mathcal{X}$ for $1 \le i \le m'$. Over rational numbers, we have

$$\begin{aligned} &H^2(\mathcal{X},\mathbb{Q}) \cong \mathbb{L}^\vee \otimes \mathbb{Q}/\textstyle\sum_{i=m'+1}^m \mathbb{Q}D_i, \\ &H_2(\mathcal{X},\mathbb{Q}) \cong \operatorname{Ker}((D_{m'+1},\dots,D_m)\colon \mathbb{L}\otimes\mathbb{Q} \to \mathbb{Q}^{m-m'}) \subset \mathbb{L}\otimes\mathbb{Q}. \end{aligned}$$

Now we introduce a canonical splitting (over $\mathbb{Q}$) of the surjection $\mathbb{L}^\vee\otimes\mathbb{Q} \to H^2(\mathcal{X},\mathbb{Q})$. For $m' < j \le m$, $b_j$ is contained in some cone in $\Sigma$ since $\Sigma$ is complete. Namely,

$$b_j = \sum_{i \notin I_j} c_{ji} b_i, \quad \text{in } \boldsymbol{N}\otimes\mathbb{Q}, \quad c_{ji} \ge 0, \quad \exists I_j \in \mathcal{A}, \tag{57}$$

where $I_j$ corresponds to the complement of the cone containing $b_j$ in its interior. By the exact sequence (56) tensored with $\mathbb{Q}$, we can find $D_j^\vee \in \mathbb{L}\otimes\mathbb{Q}$ such that

$$\langle D_i, D_j^\vee \rangle = \begin{cases} 1 & i = j \\ -c_{ji} & i \notin I_j \\ 0 & i \in I_j \setminus \{j\}. \end{cases}$$

Note that $D_j^\vee$ is uniquely determined by these conditions. These vectors $D_j^\vee$ define a decomposition

$$\mathbb{L}^\vee \otimes \mathbb{Q} = \operatorname{Ker}((D_{m'+1}^\vee,\dots,D_m^\vee)\colon \mathbb{L}^\vee\otimes\mathbb{Q} \to \mathbb{Q}^{m-m'}) \oplus \bigoplus_{j=m'+1}^m \mathbb{Q}D_j.$$

The first factor $\mathrm{Ker}(D_{m'+1}^\vee,\dots,D_m^\vee)$ is identified with $H^2(\mathcal{X},\mathbb{Q})$ under the surjection $\mathbb{L}^\vee\otimes\mathbb{Q}\to H^2(\mathcal{X},\mathbb{Q})$. Using this, we regard $H^2(\mathcal{X},\mathbb{Q})$ as a subspace of $\mathbb{L}^\vee\otimes\mathbb{Q}$. We define an *extended Kähler cone* $\tilde{C}_\mathcal{X}$ as

$$\tilde{C}_\mathcal{X}=\bigcap_{I\in\mathcal{A}}(\sum_{i\in I}\mathbb{R}_{>0}D_i)\subset\mathbb{L}^\vee\otimes\mathbb{R}.$$

Then $\eta\in\tilde{C}_\mathcal{X}$ and the image of $\eta$ in $H^2(\mathcal{X},\mathbb{R})$ is the class of the reduced symplectic form. The set $\tilde{C}_\mathcal{X}$ is the connected component of the set of regular values of the moment map $\mathfrak{h}\colon\mathbb{C}^m\to\mathbb{L}^\vee\otimes\mathbb{R}$, which contains $\eta$. The extended Kähler cone depends not only on $\mathcal{X}$ but also on the choice of our initial data. The genuine *Kähler cone* $C_\mathcal{X}$ of $\mathcal{X}$ is the image of $\tilde{C}_\mathcal{X}$ under $\mathbb{L}^\vee\otimes\mathbb{R}\to H^2(\mathcal{X},\mathbb{R})$:

$$C_\mathcal{X}=\bigcap_{I'\in\mathcal{A}'}(\sum_{i\in I'}\mathbb{R}_{>0}\overline{D}_i)\subset H^2(\mathcal{X},\mathbb{R})=H^{1,1}(\mathcal{X},\mathbb{R})$$

where $\overline{D}_i$ is the image of $D_i$ in $H^2(\mathcal{X},\mathbb{R})$. The next lemma means that the extended Kähler cone also "splits".

**Lemma 4.2.** $\tilde{C}_\mathcal{X}=C_\mathcal{X}+\sum_{j=m'+1}^m\mathbb{R}_{>0}D_j$ *in* $\mathbb{L}^\vee\otimes\mathbb{R}\cong H^2(\mathcal{X},\mathbb{R})\oplus\bigoplus_{j=m'+1}^m\mathbb{R}D_j$.

*Proof.* First note that for $1\le i\le m'$, $\overline{D}_i=D_i+\sum_{j>m'}c_{ji}D_j$, where $c_{ji}=-\langle D_i,D_j^\vee\rangle\ge 0$. Take $I'\in\mathcal{A}'$ and put $I=I'\cup\{m'+1,\dots,m\}$. It is easy to check that

$$\sum_{i\in I'}\mathbb{R}_{>0}\overline{D}_i+\sum_{j=m'+1}^m\mathbb{R}_{>0}D_j=\sum_{k\in I}\mathbb{R}_{>0}D_k\cap\bigcap_{j=m'+1}^m\{D_j^\vee>0\},$$

where we regard $D_j^\vee$ as a linear function on $\mathbb{L}^\vee\otimes\mathbb{R}$. Thus $C_\mathcal{X}+\sum_{j>m'}\mathbb{R}_{>0}D_j=\tilde{C}_\mathcal{X}\cap\bigcap_{j=m'+1}^m\{D_j^\vee>0\}$. For $j>m'$, take $I_j\in\mathcal{A}$ appearing in (57). Then $\tilde{C}_\mathcal{X}\subset\sum_{k\in I_j}\mathbb{R}_{>0}D_k\subset\{D_j^\vee>0\}$. The conclusion follows. ☐

We choose an integral basis $\{p_1,\dots,p_r\}$ of $\mathbb{L}^\vee$ such that $p_a$ is in the closure $\mathrm{cl}(\tilde{C}_\mathcal{X})$ of $\tilde{C}_\mathcal{X}$ for all $a$ and $p_{r'+1},\dots,p_r$ are in $\sum_{i=m'+1}^m\mathbb{R}_{\ge0}D_i$. Then the images $\overline{p}_1,\dots,\overline{p}_{r'}$ of $p_1,\dots,p_{r'}$ in $H^2(\mathcal{X},\mathbb{R})$ are nef and those of $p_{r'+1},\dots,p_r$ are zero. Define a matrix $(m_{ia})$ by

$$D_i=\sum_{a=1}^r m_{ia}p_a,\quad m_{ia}\in\mathbb{Z}.\tag{58}$$

Then the class $\overline{D}_i$ of the toric divisor $\{z_i=0\}$ is given by

$$\overline{D}_i=\sum_{a=1}^{r'}m_{ia}\overline{p}_a,\tag{59}$$

and $\overline{D}_j=0$ for $m'<j\le m$.

4.1.3. *Inertia components.* We introduce subsets $\mathbb{K}$, $\mathbb{K}_{\mathrm{eff}}$ of $\mathbb{L}\otimes\mathbb{Q}$ by

$$\mathbb{K}=\{d\in\mathbb{L}\otimes\mathbb{Q}\ ;\ \{i\in\{1,\dots,m\}\ ;\ \langle D_i,d\rangle\in\mathbb{Z}\}\in\mathcal{A}\},$$
$$\mathbb{K}_{\mathrm{eff}}=\{d\in\mathbb{L}\otimes\mathbb{Q}\ ;\ \{i\in\{1,\dots,m\}\ ;\ \langle D_i,d\rangle\in\mathbb{Z}_{\ge0}\}\in\mathcal{A}\}.$$

Note that $\mathbb{K}$ and $\mathbb{K}_{\rm eff}$ are not closed under addition but $\mathbb{K}$ is acted on by $\mathbb{L}$. An element of $\mathbb{K}_{\rm eff} \cap H_2(\mathcal{X},\mathbb{R})$ can be realized as a degree of a stable map of the form $\mathbb{P}(1,a) \to \mathcal{X}$. Following [9], we introduce the set Box as

$$\mathrm{Box} = \Big\{ v \in \boldsymbol{N} \ ; \ v = \sum_{k\notin I} c_k b_k \text{ in } \boldsymbol{N}\otimes\mathbb{Q}, \ c_k \in [0,1), \ I \in \mathcal{A} \Big\}.$$

For a real number $r$, let $\lceil r \rceil$, $\lfloor r \rfloor$ and $\{r\}$ denote the ceiling, floor and fractional part of $r$ respectively. For $d \in \mathbb{K}$, define an element $v(d) \in \mathrm{Box}$ by

$$v(d) := \sum_{i=1}^m \lceil \langle D_i, d\rangle \rceil b_i \in \boldsymbol{N}.$$

By the exact sequence (56), we have $v(d) = \sum_{i=1}^m \{-\langle D_i, d\rangle\} b_i$ in $\boldsymbol{N}\otimes\mathbb{Q}$ and so $v(d)$ is an element of Box. This map $d \mapsto v(d)$ factors through $\mathbb{K} \to \mathbb{K}/\mathbb{L}$ and identifies $\mathbb{K}/\mathbb{L}$ with Box. For $d \in \mathbb{K}$, we define a component $\mathcal{X}_{v(d)}$ of the inertia stack $I\mathcal{X}$ by

$$\mathcal{X}_{v(d)} = \{[z_1,\dots,z_m] \in \mathcal{X} \ ; \ z_i = 0 \text{ if } \langle D_i, d\rangle \notin \mathbb{Z}\}.$$

The stabilizer along $\mathcal{X}_{v(d)}$ is defined to be $\exp(-2\pi\sqrt{-1}d) \in \mathbb{L}\otimes\mathbb{C}^* \cong \mathbb{T}$, which acts on $\mathbb{C}^m$ by

$$(e^{-2\pi\sqrt{-1}\langle D_1, d\rangle}, \cdots, e^{-2\pi\sqrt{-1}\langle D_m, d\rangle}).$$

We can easily check that $\mathcal{X}_{v(d)}$ depends only on the element $v(d) \in \mathrm{Box}$. When $d \in \mathbb{K}_{\rm eff}\cap H_2(\mathcal{X},\mathbb{Q})$, the evaluation image of a stable map $\mathbb{P}(1,a) \to \mathcal{X}$ of degree $d$ at $[\mathbb{P}(a)] \in \mathbb{P}(1,a)$ lies on $\mathcal{X}_{\mathrm{inv}(v(d))}$. The age of $\mathcal{X}_{v(d)}$ is calculated as

$$\iota_{v(d)} := \mathrm{age}(\mathcal{X}_{v(d)}) = \sum_{i=1}^m \{-\langle D_i, d\rangle\} = \sum_{i=1}^{m'} \{-\langle D_i, d\rangle\}. \tag{60}$$

We have

$$I\mathcal{X} = \bigsqcup_{v\in\mathrm{Box}} \mathcal{X}_v, \quad H^i_{\rm orb}(\mathcal{X}) = \bigoplus_{v\in\mathrm{Box}} H^{i-2\iota_v}(\mathcal{X}_v).$$

Denote by $\mathbf{1}_v$ the unit class of $H^*(\mathcal{X}_v)$. The coarse moduli space of $\mathcal{X}_v$ is a toric variety and its cohomology ring is generated by the degree two classes $\overline{p}_1,\dots,\overline{p}_{r'}$:

$$H^*(\mathcal{X}_{v(d)}) = \big(\mathbb{C}[\overline{p}_1,\dots,\overline{p}_{r'}]/\mathfrak{J}_{v(d)}\big)\,\mathbf{1}_{v(d)}$$

where the ideal $\mathfrak{J}_{v(d)}$ is generated by $\prod_{i\in I} \overline{D}_i$ for $I \subset \{1,\dots,m\}$ such that $\{i \ ; \ \langle D_i, d\rangle \in \mathbb{Z}\} \setminus I \notin \mathcal{A}$. (See (59) for $\overline{D}_i$ in terms of $\overline{p}_a$.) Take $\xi \in \mathbb{L}^\vee$. Let $[\xi]$ be its image in $\mathbb{L}^\vee/\sum_{j=m'+1}^m \mathbb{Z}D_j \cong H^2(\mathcal{X},\mathbb{Z})$. The age $f_v([\xi]) \in [0,1)$ of the line bundle $L_\xi$ (introduced in Section 3.2) is given by

$$f_{v(d)}([\xi]) = \{-\langle \xi, d\rangle\}, \quad d \in \mathbb{K}. \tag{61}$$

4.1.4. *Weak Fano condition.* The first Chern class $\rho = c_1(T\mathcal{X}) \in H^2(\mathcal{X},\mathbb{Q})$ of $\mathcal{X}$ is the image of the vector

$$\hat{\rho} := D_1 + \cdots + D_m = \sum_{a=1}^r \rho_a p_a \in \mathbb{L}^\vee, \quad \rho_a := \sum_{i=1}^m m_{ia}.$$

$\mathcal{X}$ is *weak Fano* if $\rho$ is in the closure $\mathrm{cl}(C_{\mathcal{X}})$ of the Kähler cone $C_{\mathcal{X}}$. Later, we will use the condition that $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ which is stronger than that $\mathcal{X}$ is weak Fano.

**Lemma 4.3.** *We have* $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ *if and only if* $\rho \in \mathrm{cl}(C_{\mathcal{X}})$ *(i.e.* $\mathcal{X}$ *is weak Fano) and* $\mathrm{age}(b_j) \le 1$ *for all* $j > m'$. *Here we put* $\mathrm{age}(b_j) := \sum_{i \notin I_j} c_{ji}$. *(This coincides with* $\iota_{b_j}$ *in (60) when* $b_j \in \mathrm{Box}$. *See (57) for the definition of* $I_j$ *and* $c_{ji}$.*)*

*Proof.* From $\overline{D}_i = D_i + \sum_{j>m'} c_{ji} D_j$, we have

$$\hat{\rho} = \rho + \sum_{j>m'} (1 - \mathrm{age}(b_j)) D_j$$

The conclusion follows from Lemma 4.2. □

When $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$, we can choose a basis $p_1, \dots, p_r \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ so that $\hat{\rho}$ is in the cone generated by $p_a$'s. Thus *in this case, we will assume* $\rho_a \ge 0$ *without loss of generality.*

**Remark 4.4.** When $\mathcal{X}$ is weak Fano, we can choose our toric data so that $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ if the following condition holds for the stacky fan:

$$\{v \in \mathrm{Box}\ ;\ \mathrm{age}(v) \le 1\} \cup \{b_1, \dots, b_{m'}\} \text{ generates } \boldsymbol{N} \text{ over } \mathbb{Z}.$$

If this holds, we can choose $b_{m'+1}, \dots, b_m \in \mathrm{Box}$ so that $\{b_1, \dots, b_m\}$ generates $\boldsymbol{N}$ and $\mathrm{age}(b_j) \le 1$ for $m' < j \le m$. Then the exact sequence (56) determines $D_1, \dots, D_m$ and $\hat{\rho} = D_1 + \cdots + D_m \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ holds. That $\boldsymbol{N}$ is generated only by $b_1, \dots, b_{m'}$ is equivalent to that $\mathcal{X}$ is simply-connected in the sense of orbifold: $\pi_1^{\mathrm{orb}}(\mathcal{X}) = \{1\}$.

**Remark 4.5.** The vectors $D_j$, $m' < j \le m$ in $\mathbb{L}^\vee$ "correspond" to the following elements in the twisted sector:

$$\mathfrak{D}_j = \prod_{i \notin I_j} \overline{D}_i^{\lfloor c_{ji} \rfloor} \mathbf{1}_{v(D_j^\vee)} \in H^*_{\mathrm{orb}}(\mathcal{X}), \quad \text{where } v(D_j^\vee) = b_j + \sum_{i \notin I_j} \lceil -c_{ji} \rceil b_i. \tag{62}$$

This correspondence can be seen from the expansion of the mirror map $\tau(q)$ in Remark 4.16 below. We have $\mathfrak{D}_j = \mathbf{1}_{b_j}$ when $b_j \in \mathrm{Box}$. Therefore, if $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$ and $b_{m'+1}, \dots, b_m$ are mutually different elements in Box, we can identify $\mathbb{L}^\vee \otimes \mathbb{C}$ with the subspace $H^2(\mathcal{X}) \oplus \bigoplus_{j>m'} H^0(\mathcal{X}_{b_j})$ of $H^{\le 2}_{\mathrm{orb}}(\mathcal{X})$.

4.2. **Landau-Ginzburg model.** Following [34, 35, 40], we use the Landau-Ginzburg model as a mirror of a toric variety. By applying the exact functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to the short exact sequence (56), we have

(63)

$$\mathbf{1} \longrightarrow \mathrm{Hom}(\boldsymbol{N}, \mathbb{C}^*) \longrightarrow Y := (\mathbb{C}^*)^m \xrightarrow{\mathrm{pr}} \mathcal{M} := \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*) \longrightarrow \mathbf{1}\,.$$

The *Landau-Ginzburg model* associated to a toric orbifold is the family $\mathrm{pr}\colon Y \to \mathcal{M}$ of affine varieties given by the third arrow and a fiberwise Laurent polynomial $W\colon Y \to \mathbb{C}$, called potential, given by

$$W = w_1 + \cdots + w_m$$

where $w_1, \dots, w_m$ are the standard $\mathbb{C}^*$-valued co-ordinates on $Y = (\mathbb{C}^*)^m$. The basis of $\mathbb{L}$ dual to $p_1, \dots, p_r$ in the previous section defines $\mathbb{C}^*$-valued co-ordinates $q_1, \dots, q_r$ on $\mathcal{M} = \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$. Then the projection is given by (see (58))

$$\mathrm{pr}(w_1, \dots, w_m) = (q_1, \dots, q_r), \quad q_a = \prod_{i=1}^m w_i^{m_{ia}}. \tag{64}$$

Let $Y_q := \mathrm{pr}^{-1}(q)$ be the fiber at $q \in \mathcal{M}$ and set $W_q := W|_{Y_q}$. Note that $Y_q$ has $|\boldsymbol{N}_{\mathrm{tor}}|$ connected components and each connected component is isomorphic to $\mathrm{Hom}(\boldsymbol{N}_{\mathrm{free}}, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. Let $e_1, \dots, e_n$ be an arbitrary basis of $\boldsymbol{N}_{\mathrm{free}}$ and $y_1, \dots, y_n$ be the corresponding $\mathbb{C}^*$-valued co-ordinate on $\mathrm{Hom}(\boldsymbol{N}_{\mathrm{free}}, \mathbb{C}^*)$. We choose a splitting of the exact sequence dual to (56) over rational numbers. Namely, we take a matrix $(\ell_{ia})_{1\le i\le m, 1\le a\le r}$ with $\ell_{ia} \in \mathbb{Q}$ such that $p_a = \sum_{i=1}^m D_i \ell_{ia}$. This splitting defines a multi-valued section of $\mathrm{pr}: Y \to \mathcal{M}$ and identifies $Y_q$ with $\mathrm{Hom}(\boldsymbol{N}, \mathbb{C}^*)$. Under this identification, $y_1, \dots, y_n$ give co-ordinates on each connected component of $Y_q$ and we have

$$W|_{Y_q} = W_q = q^{\ell_1} y^{b_1} + \cdots + q^{\ell_m} y^{b_m}, \quad q^{\ell_i} = \prod_{a=1}^r q_a^{\ell_{ia}}, \quad y^{b_i} = \prod_{j=1}^n y_j^{b_{ij}}, \tag{65}$$

where $b_i = \sum_{j=1}^n b_{ij} e_j$ in $\boldsymbol{N}_{\mathrm{free}}$. Here, the choice of the branches of fractional powers of $q_a$ appearing in $q^{\ell_i}$ depends on a connected component of $Y_q$.

To proceed further, we need to restrict the parameter $q \in \mathcal{M}$ to some Zariski open subset $\mathcal{M}^{\mathrm{o}} \subset \mathcal{M}$ so that $W_q$ satisfies the "non-degeneracy condition at infinity" due to Kouchnirenko [50, 1.19].

**Definition 4.6.** Let $\hat{S}$ denote the convex hull of $b_1, \dots, b_m \in \boldsymbol{N} \otimes \mathbb{R}$. We call the Laurent polynomial $W_q(y)$ of the form (65) *non-degenerate at infinity* if for every face $\Delta$ of $\hat{S}$ (where $0 \le \dim \Delta \le n-1$), $W_{q,\Delta}(y) := \sum_{b_i \in \Delta} q^{\ell_i} y^{b_i}$ does not have critical points on $y \in (\mathbb{C}^*)^n$. Let $\mathcal{M}^{\mathrm{o}}$ be the subset of $\mathcal{M}$ consisting of $q$ for which $W_q$ is non-degenerate at infinity.

**Proposition 4.7.** *(i) Under the condition (C) in Section 4.1.1, $0 \in \boldsymbol{N} \otimes \mathbb{R}$ is in the interior of $\hat{S}$. Therefore, the Laurent polynomial $W_q$ is* convenient *in the sense of Kouchnirenko* [50, 1.5].

*(ii) $\mathcal{M}^{\mathrm{o}}$ is an open and dense subset of $\mathcal{M}$ in Zariski topology.*

*(iii) For $q \in \mathcal{M}^{\mathrm{o}}$, $W_q(y)$ has $|\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\hat{S})$ critical points on $Y_q$ (counted with multiplicities).*

*Proof.* The condition (C) implies that there exists $d \in \mathbb{L}$ such that $c_i := \langle D_i, d\rangle > 0$. Then by the exact sequence (56), we have $\sum_{i=1}^m c_i b_i = 0$. This proves (i). The statements (ii) and (iii) are due to Kouchnirenko. (ii) follows from (i) and the same argument as in [50, 6.3]. One of main theorems in [50, 1.16] states that $W_q(y)$ has $n! \operatorname{Vol}(\hat{S})$ number of critical points on each connected component of $Y_q$. (iii) follows from this and $|\pi_0(Y_q)| = |\boldsymbol{N}_{\mathrm{tor}}|$. □

Let $f_{q,z}\colon Y_q \to \mathbb{R}$ be the real part of the function $y \mapsto W_q(y)/z$. The following lemma allows us to use Morse theory for the improper function $f_{q,z}(y)$.

**Lemma 4.8.** *For each $\epsilon > 0$, the family of topological spaces*

$$\bigcup_{(q,z)\in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*} \{y \in Y_q \,;\, \|df_{q,z}(y)\| \le \epsilon\} \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$$

*is proper, i.e. pull-back of a compact set is compact. Here the norm $\|df_{q,z}(y)\|$ is taken with respect to the complete Kähler metric $\frac{1}{\sqrt{-1}} \sum_{i=1}^n d\log y_i \wedge d\overline{\log y_i}$ on $Y_q$.*

A similar statement for polynomial functions can be found in [59, Proposition 2.2 and Remarque] and this lemma may also have been well-known. We will include a proof in Appendix 7.2 since we do not know a good reference. Lemma 4.8 implies that $f_{q,z}$ satisfies the Palais-Smale condition, so that usual Morse theory applies to $f_{q,z}$ (see *e.g.* [57]). Take $(q,z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$. Since the set $\{y \in Y_q \ ; \ \|df_{q,z}(y)\| < \epsilon\}$ is compact, we can choose $M \ll 0$ so that this set is contained in $\{y \in Y_q \ ; \ f_{q,z}(y) > M\}$. Then the relative homology group $H_n(Y_q, \{y \in Y_q \ ; \ f_{q,z}(y) \le M\}; \mathbb{Z})$ is independent of the choice of such $M$ and we denote this by

$$R^\vee_{\mathbb{Z},(q,z)} = H_n(Y_q, \{y \in Y_q \ ; \ f_{q,z}(y) \ll 0\}; \mathbb{Z}), \quad (q,z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*. \tag{66}$$

The number of critical points of $f_{q,z}(y)$ is $N := |\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\hat{S})$ by Proposition 4.7. If all the critical points of $W_q(y)$ are non-degenerate, by the standard argument in Morse theory, we know that $Y_q$ is obtained from $\{f_{q,z}(y) \le M\}$ by attaching $N$ $n$-handles and so $R^\vee_{\mathbb{Z},(q,z)}$ is a free abelian group of rank $N$. If $W_q(y)$ has a critical point $y_0$ of multiplicity $\mu_0 > 1$, one can find[5] a small $C^\infty$-perturbation $\tilde{f}_{q,z}$ of $f_{q,z}$ on a small neighborhood $U_0$ of $y_0$ such that $\tilde{f}_{q,z}$ has just $\mu_0$ non-degenerate critical points in $U_0$ with Morse index $n$. By considering such a perturbation and Morse theory for $f_{q,z}$ in families (parametrized by $q$ and $z$), we obtain the following.

**Proposition 4.9.** *The relative homology groups $R^\vee_{\mathbb{Z},(q,z)}$ in (66) form a local system of rank $|\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\hat{S})$ over $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$.*

When all the critical points $\mathrm{cr}_1, \dots, \mathrm{cr}_N$ of $W_q \colon Y_q \to \mathbb{C}$ are non-degenerate, a basis of the local system $R^\vee_{\mathbb{Z}}$ is given by a set of *Lefschetz thimbles* $\Gamma_1, \dots, \Gamma_N$: the image of $\Gamma_i$ under $W_q/z$ is given by a curve $\gamma_i : [0,\infty) \to \mathbb{C}$ such that $\gamma(0) = W_q(\mathrm{cr}_i)/z$, that $\Re\gamma_i(t)$ decreases monotonically to $-\infty$ as $t \to \infty$ and that $\gamma_i$ does not pass through critical values other than $W_q(\mathrm{cr}_i)/z$; $\Gamma_i$ is the union of cycles in $W_q^{-1}(z\gamma_i(t))$ collapsing to $\mathrm{cr}_i$ along the path $\gamma_i(t)$ as $t \to 0$. When the imaginary parts $\Im(W_q(\mathrm{cr}_1)/z), \dots, \Im(W_q(\mathrm{cr}_N)/z)$ are mutually different, $\Gamma_i$ can be taken to be the union of downward gradient flowlines of $f_{q,z}(y)$ emanating from $\mathrm{cr}_i$. (Note that the gradient flow of $f_{q,z} = \Re(W_q/z)$ with respect to a Kähler metric coincides with the Hamiltonian flow generated by $\Im(W_q/z)$.) Then $\gamma_i$ becomes a half-line parallel to the real axis. The intersection pairing defines a unimodular pairing:

$$R^\vee_{\mathbb{Z},(q,-z)} \times R^\vee_{\mathbb{Z},(q,z)} \to \mathbb{Z}. \tag{67}$$

Let $R_{\mathbb{Z}} \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ be the local system dual to $R^\vee_{\mathbb{Z}}$ and $\mathcal{R} = R_{\mathbb{Z}} \otimes \mathcal{O}_{\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*}$ be the associated locally free sheaf on $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$. This inherits from the local system a Gauß-Manin connection $\widehat{\nabla}$ and a pairing $((-)^*\mathcal{R}) \otimes \mathcal{R} \to \mathcal{O}_{\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*}$.

Let $\omega_1$ be the following holomorphic volume form on $Y_1 = \operatorname{Hom}(\boldsymbol{N}, \mathbb{C}^*)$:

$$\omega_1 = \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|} \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} \quad \text{on each connected component.}$$

---

[5] We can find $\tilde{f}_{q,z}$ in the following way: Let $\rho \colon \mathbb{R}_{\ge 0} \to [0,1]$ be a $C^\infty$-function such that $\rho(r) = 1$ for $0 \le r \le 1/2$ and $\rho(r) = 0$ for $r \ge 1$. Let $U_0$ be an $\epsilon$-neighborhood of $y_0$ (in the above Kähler metric) which does not contain other critical points. Let $t = (t_1, \dots, t_n)$ be co-ordinates given by $y_i = y_{0,i} e^{t_i}$. For $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, put $f^a_{q,z}(y) = f_{q,z}(y) + \rho(|t|/\epsilon)\Re(at)$. Then for a generic, sufficiently small $a$, $\tilde{f}_{q,z} = f^a_{q,z}$ satisfies the conditions above (here, new critical points are all in $|t| < \epsilon/2$).

This is characterized as a unique translation-invariant holomorphic $n$-form $\omega_1$ satisfying $\int_{\mathrm{Hom}(\boldsymbol{N},S^1)} \omega_1 = (2\pi\sqrt{-1})^n$. By translation, $\omega_1$ defines a holomorphic volume form $\omega_q$ on each fiber $Y_q$. Let $\mathrm{pr}\colon Y^{\mathrm{o}} \to \mathcal{M}^{\mathrm{o}}$ be the restriction of the family $\mathrm{pr}\colon Y \to \mathcal{M}$ to $\mathcal{M}^{\mathrm{o}}$. Consider a relative holomorphic $n$-form of $Y^{\mathrm{o}} \times \mathbb{C}^* \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ of the form

$$\varphi = f(q,z,y) e^{W_q(y)/z}\omega_q, \quad f(q,z,y) \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}[y_1^{\pm},\dots,y_n^{\pm}] \tag{68}$$

where $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$ is the analytic structure sheaf. This relative $n$-form gives a holomorphic section $[\varphi]$ of $\mathcal{R}$ via the integration over Lefschetz thimbles:

$$\langle[\varphi],\Gamma\rangle = \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma f(q,z,y)e^{W_q(y)/z}\omega_q \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*} \tag{69}$$

The convergence of this integral is ensured by the fact that $f(q,z,y)$ has at most polynomial growth in $y$ and that $\Re(W_q(y)/z)$ goes to $-\infty$ at the end of $\Gamma$. More technically, as done in [59], one may prove the convergence of the integral by replacing the end of $\Gamma$ with a semi-algebraic chain.

Let $\mathcal{R}'$ be the $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$-submodule of $\mathcal{R}$ consisting of the sections which locally arise from relative $n$-forms $\varphi$ of the form (68). The Gauß-Manin connection on $\mathcal{R}$ preserves the subsheaf $\mathcal{R}'$. In fact, we have

$$\begin{aligned}
\widehat{\nabla}_a[\varphi] &= [(\partial_a f + \frac{1}{z}(\partial_a W_q) f)e^{W_q/z}\omega_q],\\
\widehat{\nabla}_{z\partial_z}[\varphi] &= [(z\partial_z f - \frac{1}{z}W_q f - \frac{n}{2}f)e^{W_q/z}\omega_q],
\end{aligned}$$

where $\varphi$ is given in (68) and $\partial_a = q_a(\partial/\partial q_a)$. Take a generic $q \in \mathcal{M}^{\mathrm{o}}$ such that all the critical points of $W_q(y)$ are non-degenerate. Let $\Gamma_1,\dots,\Gamma_N$ be Lefschetz thimbles of $W_q(y)/z$ corresponding to critical points $\mathrm{cr}_1,\dots,\mathrm{cr}_N$. Then we have the following asymptotic expansion as $z \to 0$ with $\arg(z)$ fixed:

$$\frac{1}{(-2\pi z)^{n/2}}\int_{\Gamma_i} f(q,z,y)e^{W_q(y)/z}\omega_q \sim \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|}\frac{e^{W_q(\mathrm{cr}_i)/z}}{\sqrt{\mathrm{Hess}(W_q)(\mathrm{cr}_i)}}(f(q,0,\mathrm{cr}_i)+O(z)) \tag{70}$$

where $f(q,z,y) \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}[y_1^{\pm},\dots,y_n^{\pm}]$ is regular at $z=0$ and $\mathrm{Hess}(W_q)$ is the Hessian of $W_q$ calculated in co-ordinates $\log y_1,\dots,\log y_n$. Let $J(W_q)$ be the Jacobi ring of $W_q$:

$$J(W_q) := \mathbb{C}[Y_q]\Big/\langle \frac{\partial W_q}{\partial y_1},\dots,\frac{\partial W_q}{\partial y_n}\rangle,$$

where $\mathbb{C}[Y_q]$ is the co-ordinate ring of $Y_q$. Let $\phi_i(y) \in \mathbb{C}[Y_q]$ be a function which represents a delta-function supported on $\mathrm{cr}_i$ in the Jacobi ring $J(W_q)$. Put $\varphi_i = \phi_i(y)e^{W_q/z}\omega_q$. By the asymptotics of $\langle[\varphi_i],\Gamma_j\rangle$, we know that $[\varphi_1],\dots,[\varphi_N]$ form a basis of $\mathcal{R}$ for sufficiently small $|z|>0$. Since $\mathcal{R}'$ is preserved by the Gauß-Manin connection, we have $\mathcal{R}=\mathcal{R}'$ on the whole $\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*$. In other words, $\mathcal{R}$ is generated by relative $n$-forms of the form (68). Let $\Gamma_1^\vee,\dots,\Gamma_N^\vee$ be Lefschetz thimbles of $W_q/(-z)$ dual to $\Gamma_1,\dots,\Gamma_N$ with respect to the intersection pairing (67). Then the pairing on $\mathcal{R}$ can be written as

$$([\varphi(-z)],[\varphi'(z)])_{\mathcal{R}} = \frac{1}{(2\pi\sqrt{-1}z)^n}\sum_{i=1}^N \int_{\Gamma_i^\vee}\varphi(-z)\cdot\int_{\Gamma_i}\varphi'(z). \tag{71}$$

We define an extension $\mathcal{R}^{(0)}$ of $\mathcal{R}$ to $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}$ as follows: a section of $\mathcal{R}$ on an open set $U \times \{0 < |z| < \epsilon\}$ is defined to be extendible to $z = 0$ if it is the image of a relative $n$-form $s$ of the form (68) such that $f(q, z, y)$ in (68) is regular at $z = 0$. When $[\varphi]$ and $[\varphi']$ are extendible to $z = 0$, we have from (71) and (70)

$$([\varphi], [\varphi'])_{\mathcal{R}} \sim \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|^2} \sum_{i=1}^{N} \frac{f(q, 0, \mathrm{cr}_i) f'(q, 0, \mathrm{cr}_i)}{\operatorname{Hess} W_q(\mathrm{cr}_i)} + O(z)$$

where we put $\varphi = f(q, z, y) e^{W_q(y)/z} \omega_q$ and $\varphi' = f'(q, z, y) e^{W_q(y)/z} \omega_q$. This shows that $([\varphi], [\varphi'])_{\mathcal{R}}$ is regular at $z = 0$ and the value at $z = 0$ equals the residue pairing on $J(W_q)$. By continuity, we have at all $q \in \mathcal{M}^{\mathrm{o}}$:

$$([\varphi], [\varphi'])_{\mathcal{R}}|_{z=0} = \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|^2} \operatorname{Res}_{Y^{\mathrm{o}}/\mathcal{M}^{\mathrm{o}}} \left[ \frac{f(q, 0, y) f'(q, 0, y) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}}{y_1 \frac{\partial W_q}{\partial y_1}, \dots, y_n \frac{\partial W_q}{\partial y_n}} \right].$$

Let $\phi'_1, \dots, \phi'_N$ be an arbitrary basis of the Jacobi ring and put $s_i := [\phi'_i(y) e^{W_q(y)/z} \omega_q]$. Then the Gram matrix $(s_i, s_j)_{\mathcal{R}}$ is non-degenerate in a neighborhood of $z = 0$ since the residue pairing is non-degenerate. This implies that $s_1, \dots, s_N$ form a local basis of $\mathcal{R}^{(0)}$ around $z = 0$. Summarizing,

**Proposition 4.10** ([25, Lemma 2.21])**.** *The $\mathcal{O}_{\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*}$-module $\mathcal{R}$ is generated by relative $n$-forms of the form (68). The extension $\mathcal{R}^{(0)}$ of $\mathcal{R}$ to $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}$ is locally free and the pairing on $\mathcal{R}$ extends to a non-degenerate pairing $((-)^* \mathcal{R}^{(0)}) \otimes \mathcal{R}^{(0)} \to \mathcal{O}_{\mathcal{M}^{\mathrm{o}} \times \mathbb{C}}$.*

Sabbah [62] gave a different construction of $\mathcal{R}^{(0)}$ based on the algebraic Gauß-Manin system and the Fourier-Laplace transformation. The corresponding results were shown in [62, Corollary 10.2].

We introduce the *Euler vector field* $E$ on $\mathcal{M}^{\mathrm{o}}$ by

$$E := \mathrm{pr}_* \left( \sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i} \right) = \sum_{a=1}^{r} \rho_a q_a \frac{\partial}{\partial q_a}, \quad \rho_a = \sum_{i=1}^{m} m_{ia}.$$

The grading operator $\mathsf{Gr}$ acting on sections of $\mathcal{R}^{(0)}$ is defined by

$$\mathsf{Gr}[\varphi] = 2 \left[ \left( z \frac{\partial f}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial f}{\partial w_i} \right) e^{W/z} \omega \right] \tag{72}$$

for a section $[\varphi]$ of the form (68). This grading operator can be written in terms of the Gauß-Manin connection and the Euler vector field (*c.f.* (5)):

**Lemma 4.11.** $\mathsf{Gr} = 2(\widehat{\nabla}_E + \widehat{\nabla}_{z\partial_z} + \frac{n}{2})$.

*Proof.* By the multi-valued splitting of the fibration (63) appearing in the beginning of this section, we can regard $E$ as a vector field on $Y$. Using the co-ordinate system $(q_a, y_i)$ associated to this splitting, we write $\sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i} = E + \sum_{i=1}^{n} c_i y_i \frac{\partial}{\partial y_i}$ for some $c_i \in \mathbb{Q}$. Because $(\sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i}) W = W$, we have

$$\begin{aligned} \frac{1}{2} \mathsf{Gr}[\varphi] &= \left[ \left( (z\partial_z + \textstyle\sum_{i=1}^{m} w_i \partial_{w_i}) (f e^{W/z}) \right) \omega \right] \\ &= \left( \widehat{\nabla}_{z\partial_z} + \frac{n}{2} + \widehat{\nabla}_E \right) [\varphi] + \left[ \left( (\textstyle\sum_{i=1}^{n} c_i y_i \partial_{y_i}) (f e^{W/z}) \right) \omega \right]. \end{aligned}$$

The second term is zero in cohomology since it is exact. □

**Definition 4.12.** Let $\pi\colon \mathcal{M}^{\mathrm{o}}\times\mathbb{C}\to\mathcal{M}^{\mathrm{o}}$ be the projection. The *B-model* $\frac{\infty}{2}$ *VHS of the Landau-Ginzburg model* is a locally free $\pi_*\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}$-module $\mathcal{F}:=\pi_*\mathcal{R}^{(0)}$ endowed with a flat connection $\nabla\colon\mathcal{F}\to z^{-1}\mathcal{F}\otimes\Omega^1_{\mathcal{M}}$ induced from the Gauß-Manin connection $\widehat{\nabla}$, a pairing $(\cdot,\cdot)_{\mathcal{F}}\colon\mathcal{F}\times\mathcal{F}\to\pi_*\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$ in (71) induced from the intersection pairing (67) and a grading operator $\mathsf{Gr}\colon\mathcal{F}\to\mathcal{F}$ in (72). The B-model $\frac{\infty}{2}$VHS has a natural integral structure given by the local system $R_{\mathbb{Z}}$ of relative cohomology groups on $\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*$.

4.3. **Mirror symmetry for toric orbifolds.** We explain a version of mirror symmetry conjecture for weak Fano toric orbifolds, which we assume in the rest of the paper. This has been proved for weak Fano toric manifolds [35] and weighted projective spaces [23]. A general case for toric orbifolds will be proved in [22].

Mirror symmetry roughly states that the B-model $\frac{\infty}{2}$VHS in the previous section is isomorphic to the A-model $\frac{\infty}{2}$VHS restricted to a certain subspace (basically $H^{\le 2}_{\mathrm{orb}}(\mathcal{X})$) of $H^*_{\mathrm{orb}}(\mathcal{X})$ under a suitable identification of the base space. This isomorphism is given by a so called *I-function* which is a cohomology-valued function on an open domain of the base space $\mathcal{M}^{\mathrm{o}}$ of the B-model $\frac{\infty}{2}$VHS.

**Definition 4.13** ([22])**.** The $I$-function for a projective toric orbifold $\mathcal{X}$ is a cohomology-valued power series on $\mathcal{M}$ defined by

$$I(q,z)=e^{\sum_{a=1}^r\overline{p}_a\log q_a/z}\sum_{d\in\mathbb{K}_{\mathrm{eff}}}q^d\frac{\prod_{i;\langle D_i,d\rangle<0}\prod_{\langle D_i,d\rangle\le\nu<0}(\overline{D}_i+(\langle D_i,d\rangle-\nu)z)}{\prod_{i;\langle D_i,d\rangle>0}\prod_{0\le\nu<\langle D_i,d\rangle}(\overline{D}_i+(\langle D_i,d\rangle-\nu)z)}\mathbf{1}_{v(d)}$$

where $q^d=q_1^{\langle p_1,d\rangle}\dots q_r^{\langle p_r,d\rangle}$ for $d\in\mathbb{L}\otimes\mathbb{Q}$, the index $\nu$ moves in $\mathbb{Z}$. Recall that $\overline{p}_a$ and $\overline{D}_j$ are images of $p_a$ and $D_j$ under the projection $\mathbb{L}^\vee\otimes\mathbb{Q}\to H^2(\mathcal{X},\mathbb{Q})$. Note that $\overline{p}_a=0$ for $a>r'$, $\overline{D}_j=0$ for $j>m'$ and $\langle p_a,d\rangle\ge 0$ for $d\in\mathbb{K}_{\mathrm{eff}}$.

The following lemma is easy to check.

**Lemma 4.14.** *The $I$-function is a convergent power series in $q_1,\dots,q_r$ if and only if $\hat{\rho}$ is in the closure* $\mathrm{cl}(\tilde{C}_{\mathcal{X}})$ *of the extended Kähler cone. In this case, the $I$-function has the asymptotics*

$$I(q,z)=1+\frac{\tau(q)}{z}+o(z^{-1})$$

*where $\tau(q)$ is the multi-valued function taking values in $H^{\le 2}_{\mathrm{orb}}(\mathcal{X})$.*

To state mirror symmetry conjecture, we need to assume that $\hat{\rho}\in\mathrm{cl}(\tilde{C}_{\mathcal{X}})$. See Section 4.1.4 for this condition. The $I$-function is multi-valued and the fundamental group $\pi_1(\mathcal{M})\cong\mathbb{L}^\vee$ acts on it by monodromy transformations. Take a loop $t\mapsto e^{-2\pi\sqrt{-1}\xi t}q=(e^{-2\pi\sqrt{-1}\xi_1 t}q_1,\dots,e^{-2\pi\sqrt{-1}\xi_r t}q_r)$ for $\xi=\sum_{a=1}^r\xi_a p_a\in\mathbb{L}^\vee$. The monodromy of $I(q,z)$ along this loop is given by

$$I(e^{-2\pi\sqrt{-1}\xi}q,z)=G^{\mathcal{H}}([\xi])I(q,z)$$

where $G^{\mathcal{H}}([\xi])$ is the Galois action (31) corresponding to $[\xi]\in\mathbb{L}^\vee/\sum_{j>m'}\mathbb{Z}D_j\cong H^2(\mathcal{X},\mathbb{Z})$. Therefore, we have

$$\tau(e^{-2\pi\sqrt{-1}\xi}q)=G([\xi])\tau(q)$$

where $\tau(q)$ is a function in Lemma 4.14 and $G([\xi])$ is given in (24). This shows that $\tau(q)$ induces a single-valued map

$$\tau\colon (\text{neighborhood of } q=0 \text{ in } \mathcal{M}) \longrightarrow H^{\le 2}_{\mathrm{orb}}(\mathcal{X})/H^2(\mathcal{X},\mathbb{Z}).$$

Let $\mathcal{F}_{\mathrm{B}} \to \mathcal{M}^{\mathrm{o}}$ be the B-model $\frac{\infty}{2}$VHS associated with Landau-Ginzburg model mirror to $\mathcal{X}$ and $\mathcal{F}_{\mathrm{A}} = (\widetilde{\mathcal{F}}_{\mathrm{A}} \to U)/H^2(\mathcal{X},\mathbb{Z})$ be the A-model $\frac{\infty}{2}$VHS of $\mathcal{X}$, where $U$ is a suitable open domain in $H^*_{\mathrm{orb}}(\mathcal{X})$ (see Section 3).

**Conjecture 4.15.** *Assume that our toric data satisfy $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$. There exists an isomorphism of graded $\frac{\infty}{2}$VHS* $\mathrm{Mir}\colon \mathcal{F}_{\mathrm{B}} \cong \tau^*\mathcal{F}_{\mathrm{A}}$. *This isomorphism sends the section $[e^{W_q/z}\omega_q]$ of $\mathcal{F}_{\mathrm{B}}$ to the $I$-function $I(q,z) \in \mathcal{H}^{\mathcal{X}}$, i.e.*

$$\mathcal{J}_{\tau(q)}(\mathrm{Mir}[e^{W_q/z}\omega_q]) = I(q,z)$$

*where $\mathcal{J}_{\tau(q)}\colon \widetilde{\mathcal{F}}_{\mathrm{A},\tau(q)} \to \mathcal{H}^{\mathcal{X}}$ is the embedding (28) given by the fundamental solution.*

**Remark 4.16.** (i) We have $\operatorname{rank}\mathcal{F}_{\mathrm{A}} = \dim H^*_{\mathrm{orb}}(\mathcal{X})$ and $\operatorname{rank}\mathcal{F}_{\mathrm{B}} = |\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\hat{S})$. These two numbers match if and only if $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$.

(ii) The map $\tau$ takes the form

$$\tau(q) = \sum_{a=1}^{r'} (\log q_a) p_a + \sum_{j=m'+1}^{m} q^{D_j^\vee}\mathfrak{D}_j + \text{ higher terms.}$$

Thus $\tau$ is a local embedding (isomorphism) near $q=0$ if $p_1,\dots,p_{r'},\mathfrak{D}_{m'+1},\dots,\mathfrak{D}_m$ are linearly independent (resp. basis of $H^{\le 2}_{\mathrm{orb}}(\mathcal{X})$). See (62) for $\mathfrak{D}_j$.

(iii) Because of the asymptotic of the $I$-function in Lemma 4.14, the conjecture further implies that $\mathrm{Mir}[e^{W_q/z}\omega_q]$ is the unit section $\mathbf{1}$ of the A-model $\frac{\infty}{2}$VHS $\mathcal{F}_{\mathrm{A}}$. This, however, fails to hold for non-weak-Fano case [44].

As in the previous section, we denote by $\mathcal{R}$ the locally free $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$-module associated to the local system $R$ of the Landau-Ginzburg model. We regard $[e^{W_q/z}\omega_q]$ as a section of $\mathcal{R}$. Via the identification (32) of $\mathcal{V}^{\mathcal{X}} = H^*_{\mathrm{orb}}(\mathcal{X})$ with the quantum cohomology local system, the mirror map Mir induces an isomorphism

$$\mathrm{Mir}_{(q,z)}\colon \mathcal{R}_{(q,z)} \to \mathcal{V}^{\mathcal{X}} \quad \text{s.t. } \mathrm{Mir}_{(q,z)}([e^{W_q/z}\omega_q]) = z^{-\rho}z^{\mu}I(q,z), \tag{73}$$

at $(q,z) \in (\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*)^\sim$. We will compute integral linear co-ordinates on $\mathcal{V}^{\mathcal{X}}$ corresponding to Lefschetz thimbles $\Gamma_k \in R^\vee_{\mathbb{Z}}$ through this map $\mathrm{Mir}_{(q,z)}$.

Now we can state the main theorem.

**Theorem 4.17.** *Let $\mathcal{X}$ be a weak Fano toric orbifold defined by initial data satisfying $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$. Assume that Conjecture 4.15 and (A3) in Section 3.5 hold for $\mathcal{X}$. Then via the mirror isomorphism, the integral structure in Landau-Ginzburg B-model induces the $\widehat{\Gamma}$-integral structure of $\mathcal{X}$ in Definition-Proposition 3.16.*

The next section is devoted to the proof of this theorem.

**Remark 4.18.** Since the $\widehat{\Gamma}$-integral structure is defined to be the image of the $K$-group, we can identify the integral lattice $R^\vee_{\mathbb{Z},(q,z)}$ generated by Lefschetz thimbles with (the dual of) the $K$-group $K(\mathcal{X})$ by Theorem 4.17. This correspondence also identifies the intersection numbers of vanishing cycles with the Mukai pairing on the $K$-group.

On the other hand, by mirror symmetry again, the Stokes matrices of the quantum differential equations at the irregular singular point $z = 0$ are known to arise as the intersection numbers of vanishing cycles. In particular, there exist $V_1, \dots, V_N \in K(\mathcal{X})$ (which correspond to a set of Lefschetz thimbles) such that the Stokes matrix is given by a matrix of the Mukai pairing $(V_i, V_j)_{K(\mathcal{X})}$. Dubrovin's conjecture [31] says that $V_1, \dots, V_N$ here should come from an exceptional collection in the derived category. This should follow from homological mirror symmetry for toric orbifolds. Several versions of homological mirror symmetry for toric manifolds were proved by Abouzaid [1], Fang-Liu-Treumann-Zaslow [32].

4.4. **Oscillatory integrals.** The proof of Theorem 4.17 is based on a calculation of oscillatory integrals $\int_{\Gamma_0} e^{W_q/z}\omega_q$ for some special Lefschetz thimble $\Gamma_0$. We will make use of Givental's *equivariant mirror* which gives a perturbation of oscillatory integrals. This is considered to be the mirror of the equivariant quantum cohomology of toric orbifolds, although we do not give a precise formulation of equivariant mirror symmetry.

4.4.1. *Equivariant mirror.* Let $T := (\mathbb{C}^*)^m$ act on our toric orbifold $\mathcal{X} = \mathbb{C}^m/\!/\mathbb{T}$ via the diagonal action of $(\mathbb{C}^*)^m$ on $\mathbb{C}^m$. Let $-\lambda_1, \dots, -\lambda_m$ be the equivariant variables corresponding to generators of $H_T^*(\mathrm{pt})$. We will regard $\lambda_i$ either as a cohomology class or as a complex number depending on the context. Givental's equivariant mirror [35] is given by the following perturbed potential $W^\lambda$:

$$W^\lambda := \sum_{i=1}^m (w_i + \lambda_i \log w_i) = W + \sum_{i=1}^m \lambda_i \log w_i$$

Here $\lambda_i$ denotes a complex number. This is a multi-valued function on each fiber $Y_q$. Morse theory for $\Re(W^\lambda(y)/z)$ will compute relative homology with coefficients in some local system. For a cycle $\Gamma \subset Y_q$ in such a relative homology, we can define the *equivariant oscillatory integral*:

$$\int_\Gamma e^{W^\lambda/z}\omega_q = \int_\Gamma e^{W/z} \prod_{i=1}^m w_i^{\lambda_i/z}\omega_q.$$

For our purpose, it is more convenient to use the exponent $\lambda_i/(2\pi\sqrt{-1})$ instead of $\lambda_i/z$. Define

$$\mathcal{I}_\Gamma^\lambda(q,z) := \frac{1}{(-2\pi z)^{n/2}} \int_\Gamma e^{\frac{w_1+\cdots+w_m}{z}} \prod_{i=1}^m w_i^{\frac{\lambda_i}{2\pi\sqrt{-1}}}\omega_q. \tag{74}$$

Consider the fibration formed by real points on (63):

$$\mathbf{1} \longrightarrow \mathrm{Hom}(\boldsymbol{N}, \mathbb{R}_{>0}) \longrightarrow Y_\mathbb{R} := \mathbb{R}_{>0}^m \xrightarrow{\mathrm{pr}|_{Y_\mathbb{R}}} \mathcal{M}_\mathbb{R} := \mathrm{Hom}(\mathbb{L}, \mathbb{R}_{>0}) \longrightarrow \mathbf{1}\,.$$

Here we regard $\mathbb{R}_{>0}$ as an abelian group with respect to the multiplication. Note that this exact sequence splits and that the section given by the matrix $(\ell_{ia})$ in Section 4.2 is single-valued when restricted to this real locus. Take a point $q \in \mathcal{M}_\mathbb{R}^\mathrm{o} = \mathcal{M}_\mathbb{R} \cap \mathcal{M}^\mathrm{o}$. Let $\Gamma_0 := Y_q \cap Y_\mathbb{R} \cong \mathrm{Hom}(\boldsymbol{N}, \mathbb{R}_{>0})$ be a cycle formed by real points in the fiber $Y_q$. In co-ordinates $y_1, \dots, y_n$ in Section 4.2, we have $\Gamma_0 = \{(y_1, \dots, y_n) \in Y_q \ ; \ y_i \in \mathbb{R}_{>0}\}$. Then the integral $\mathcal{I}_{\Gamma_0}^\lambda(q,z)$ is well-defined when $q \in \mathcal{M}_\mathbb{R}^\mathrm{o}$ and $\Re(z) < 0$.

4.4.2. *$H$-function.* It is convenient to introduce another cohomology-valued hypergeometric function $H(q,z)$, which is related to $I(q,z)$ by a $q$-independent linear transformation. This has been used by Horja [42], Hosono [43] and Borisov-Horja [11] in the context of homological mirror symmetry. Using Gamma functions, we can write

$$I(q,z) = e^{\sum_{a=1}^r \overline{p}_a \log q_a/z} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} \frac{q^d}{z^{\langle\hat{\rho},d\rangle}} \prod_{i=1}^m \frac{\Gamma(1-\{-\langle D_i,d\rangle\}+\overline{D}_i/z)}{\Gamma(1+\langle D_i,d\rangle+\overline{D}_i/z)} \frac{\mathbf{1}_{v(d)}}{z^{\iota_{v(d)}}}.$$

Via the identification $z^{-\rho}z^{\mu}\colon \mathsf{H}_z^{\mathcal{X}} \cong \mathcal{V}^{\mathcal{X}}$ in (32), the $I$-function gives a $\mathcal{V}^{\mathcal{X}}$-valued function

$$z^{-\rho}z^{\mu}I(q,z) = z^{-n/2} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} x^{\overline{p}+d} \prod_{i=1}^m \frac{\Gamma(1-\{-\langle D_i,d\rangle\}+\overline{D}_i)}{\Gamma(1+\langle D_i,d\rangle+\overline{D}_i)} \mathbf{1}_{v(d)}$$

where we used the notation:

$$x^{\overline{p}+d} := e^{\sum_{a=1}^r \overline{p}_a \log x_a} \prod_{a=1}^r x_a^{\langle p_a,d\rangle}, \quad x_a := \frac{q_a}{z^{\rho_a}}.$$

We can decompose the map $\Psi\colon K(\mathcal{X}) \to \mathcal{V}^{\mathcal{X}}$ in (54) as

$$\Psi\colon K(\mathcal{X}) \xrightarrow{\widetilde{\mathrm{ch}}} H^*(I\mathcal{X}) \xrightarrow{(2\pi)^{-n/2}\widehat{\Gamma}_{\mathcal{X}}(2\pi\sqrt{-1})^{\deg/2}\mathrm{inv}^*} \mathcal{V}^{\mathcal{X}}.$$

The $H$-function is defined to be a function which takes values in the middle vector space $H^*(I\mathcal{X})$ and corresponds to $z^{-\rho}z^{\mu}I(q,z)$ via the second map above:

$$\begin{aligned} H(q,z) &:= (2\pi)^{\frac{n}{2}} \mathrm{inv}^*(2\pi\sqrt{-1})^{-\frac{\deg}{2}} \widehat{\Gamma}_{\mathcal{X}}^{-1}(z^{-\rho}z^{\mu}I(q,z)) \\ &= (2\pi)^{\frac{n}{2}} z^{-\frac{n}{2}} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} x^{\frac{\overline{p}}{2\pi\sqrt{-1}}+d} \frac{\mathbf{1}_{\mathrm{inv}(v(d))}}{\prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle + \frac{\overline{D}_i}{2\pi\sqrt{-1}})}. \end{aligned}$$

Here we used that $\widehat{\Gamma}_{\mathcal{X}}^{-1}$ cancels exactly with the numerator in the formula of $z^{-\rho}z^{\mu}I(q,z)$.

The $I$-function and the $H$-function admit equivariant generalizations. An element $\xi \in \mathbb{L}^\vee$ defines a $T$-equivariant orbifold line bundle $L_\xi$ on $\mathcal{X}$:

$$L_\xi = \mathcal{U}_\eta \times \mathbb{C} \Big/ (z_1,\dots,z_m,c) \sim (t^{D_1}z_1,\dots,t^{D_m}z_m,t^{\xi}c), \ t\in\mathbb{T}$$

where $T=(\mathbb{C}^*)^m$ acts on $L_\xi$ by the diagonal action on the first factor and the trivial action on the second factor. By taking the equivariant first Chern class, we can identify $\xi\in\mathbb{L}^\vee$ with $c_1^T(L_\xi)$ of $H_T^2(\mathcal{X})$. By abuse of notation, we denote by $\overline{p}_1,\dots,\overline{p}_r \in H_T^2(\mathcal{X})$ the $T$-*equivariant* cohomology classes corresponding to $p_1,\dots,p_r\in\mathbb{L}^\vee$. Note that $\overline{p}_{r'+1},\dots,\overline{p}_r$ are non-zero only in equivariant cohomology. Similarly, we denote by $\overline{D}_i \in H_T^2(\mathcal{X})$ the $T$-equivariant Poincaré dual of the toric divisor $\{z_i=0\}$. Note that $\overline{D}_j = 0$ for $j>m'$ even in equivariant cohomology (since $\{z_j=0\}$ is empty). Then we have (*c.f.* Equation (58))

$$\overline{D}_i = \sum_{a=1}^r m_{ia}\overline{p}_a - \lambda_i \quad \text{in } H_T^2(\mathcal{X}). \tag{75}$$

The equivariant $I$-function is defined by the same formula in Definition 4.13 with all the appearance of $\overline{p}_a$ and $\overline{D}_j$ replaced by their equivariant counterparts. The equivariant

$H$-function $H^\lambda(q,z)$ is a slight modification of $H(q,z)$:

$$H^\lambda(q,z) := (2\pi)^{\frac{n}{2}} z^{-\frac{n}{2}+\frac{\lambda_1+\cdots+\lambda_m}{2\pi\sqrt{-1}}} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} x^{\frac{\overline{p}}{2\pi\sqrt{-1}}+d} \frac{\mathbf{1}_{\mathrm{inv}(v(d))}}{\prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle + \frac{\overline{D}_i}{2\pi\sqrt{-1}})}, \tag{76}$$

where $\overline{p}_a, \overline{D}_i \in H^2_T(\mathcal{X})$. The equivariant $I$- and $H$-functions take values in $H^*_{\mathrm{orb},T}(\mathcal{X})$ and $H^*_T(I\mathcal{X})$ respectively (here $H^*_{\mathrm{orb},T}(\mathcal{X})$ is $H^*_T(I\mathcal{X})$ with a different grading).

4.4.3. *Oscillatory integral and $H$-function.* We will show the following:

**Theorem 4.19.** *Assume that $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$. The equivariant oscillatory integral (74) and the equivariant $H$-function (76) are related by*

$$\mathcal{I}^\lambda_{\Gamma_0}(q,z) = \int_{I\mathcal{X}} H^\lambda(q,z)\,\mathrm{Td}^\lambda_{\mathcal{X}}, \quad q \in \mathcal{M}^{\mathrm{o}}_{\mathbb{R}},\ z<0. \tag{77}$$

*where $\mathrm{Td}^\lambda_{\mathcal{X}}$ is the $T$-equivariant Todd class. The branch of the $H$-function is chosen so that $\Im \log z = \pi, \Im\log q_a = 0$. In the non-equivariant limit, we have*

$$\langle [e^{W_q/z}\omega_q], \Gamma_0\rangle = (\widetilde{\mathrm{ch}}^{-1} H(q,z), \mathcal{O}_{\mathcal{X}})_{K(\mathcal{X})_{\mathbb{C}}}, \tag{78}$$

*where the left-hand side is the pairing in (69) and the right-hand side is the complexified Mukai pairing. Therefore, under the mirror isomorphism (73) and $\Psi^{-1}\colon \mathcal{V}^{\mathcal{X}} \cong K(\mathcal{X})\otimes\mathbb{C}$, the real thimble $\Gamma_0 \in \mathcal{R}^\vee_{\mathbb{Z}}$ corresponds to the linear form $(\cdot, \mathcal{O}_{\mathcal{X}})_{K(\mathcal{X})_{\mathbb{C}}}$ on $K(\mathcal{X})$.*

**Remark 4.20.** (i) Even if $\hat{\rho} \notin \mathrm{cl}(\tilde{C}_{\mathcal{X}})$, the left-hand side of (77) makes sense as an analytic function in $q$ and $z$. In this case, the right-hand side could be understood as the asymptotic expansion in $q_1,\dots,q_r$ of the left-hand side in the limit $q_a \searrow +0$.

(ii) The relation (77) gives a connection between solutions to ordinary differential equations in $z$. Both hand sides satisfy the same differential equations in $z$ (see below). The oscillatory integral admits an asymptotic expansion in $z$ and the $H$-function is by definition a power series in $z^{-1}$ (when $\hat{\rho} \in \mathrm{cl}(\tilde{C}_{\mathcal{X}})$).

(iii) This theorem suggests that, under homological mirror symmetry, the thimble $\Gamma_0$, an object of Fukaya-Seidel category of the Landau-Ginzburg model, should correspond to the structure sheaf $\mathcal{O}_{\mathcal{X}}$, an object of the derived category of coherent sheaves on $\mathcal{X}$. This correspondence is consistent with the SYZ picture. The Lefschetz thimble $\Gamma_0$ gives a Lagrangian section of the SYZ fibration, so should correspond to the structure sheaf.

By the localization theorem [5] in equivariant cohomology, the inclusion $i\colon I\mathcal{X}^T \to I\mathcal{X}$ induces an isomorphism $i^*\colon H^*_T(I\mathcal{X}) \otimes_{H^*_T(\mathrm{pt})} \mathbb{C}(\lambda) \to H^*(I\mathcal{X}^T)\otimes\mathbb{C}(\lambda)$, where $I\mathcal{X}^T$ is the set of $T$-fixed points in $I\mathcal{X}$ and $\mathbb{C}(\lambda)$ is the fraction ring of $H^*_T(\mathrm{pt}) = \mathbb{C}[\lambda_1,\dots,\lambda_m]$. For the case of toric orbifolds, the number of fixed points in $I\mathcal{X}$ is equal to $N = \dim H^*_{\mathrm{orb}}(\mathcal{X})$. A $T$-fixed point in $I\mathcal{X}$ is labeled by a pair $(\sigma, v)$ of a fixed point $\sigma \in \mathcal{X}^T$ and $v \in \mathrm{Box}$ such that $\sigma \in \mathcal{X}_v$. Note that a fixed point $\sigma \in \mathcal{X}$ is in one-to-one correspondence with a top dimensional cone of the fan $\Sigma$ spanned by $\{b_i\ ;\ \sigma \in \{z_i = 0\}\}$. By restricting $H^\lambda(q,z)$ to a fixed point $(\sigma,v)$, we get a function $H^\lambda_{\sigma,v}(q,z)$ in $q$, $z$ and $\lambda$. We call it a *component* of the $H$-function.

**Lemma 4.21.** *The equivariant $H$-function $H^\lambda(q,z)$ and the oscillatory integral $\mathcal{I}^\lambda_{\Gamma_0}(q,z)$ are solutions to the following GKZ-type differential equations:*

$$\left[\prod_{\langle D_i,d\rangle>0}\prod_{\nu=0}^{\langle D_i,d\rangle-1}(\hat{D}^\lambda_i-\nu z)-q^d\prod_{\langle D_i,d\rangle<0}\prod_{\nu=0}^{-\langle D_i,d\rangle-1}(\hat{D}^\lambda_i-\nu z)\right]f=0,\quad d\in\mathbb{L},$$

$$\left(z\partial_z+\sum_{a=1}^r\rho_a\partial_a-\frac{\lambda_1+\cdots+\lambda_m}{2\pi\sqrt{-1}}+\frac{n}{2}\right)f=0,$$

*where $\hat{D}^\lambda_i=z\sum_{a=1}^r m_{ia}\partial_a-z\lambda_i/(2\pi\sqrt{-1})$ and $\partial_a=q_a(\partial/\partial q_a)$. Note that $\langle D_i,d\rangle\in\mathbb{Z}$ for $d\in\mathbb{L}$. The $N$ components $H^\lambda_{\sigma,v}(q,z)$ of the $H$-function form a basis of solutions to these differential equations for a generic $\lambda$.*

From this lemma[6], we know that there exist coefficient functions $c_{\sigma,v}(\lambda)$ such that

$$\mathcal{I}^\lambda_{\Gamma_0}(q,z)=\sum_{(\sigma,v)\in I\mathcal{X}^T}c_{\sigma,v}(\lambda)H^\lambda_{\sigma,v}(q,z). \tag{79}$$

We will determine a holomorphic function $c_{\sigma,v}(\lambda)$ in $\lambda$ by putting $z=-1$ and studying the asymptotic behavior of the both hand sides in the limit $q_a\searrow+0$. Take a fixed point $\sigma\in\mathcal{X}^T$. Define $I^\sigma\in\mathcal{A}$ by $I^\sigma=\{i\ ;\ \sigma\notin\{z_i=0\}\}$. We can take $\{w_j\ ;\ j\notin I^\sigma\}$ as a co-ordinate system on $Y_q\cap Y_\mathbb{R}=\Gamma_0$. We can express $w_i$ for $i\in I^\sigma$ in terms of $\{w_j\ ;\ j\notin I^\sigma\}$ and $q_a$, $a=1,\dots,r$ by solving (64). Put

$$w_i=\prod_{a=1}^r q_a^{\ell^\sigma_{ia}}\prod_{j\notin I^\sigma}w_j^{b^\sigma_{ij}},\quad i\in I^\sigma.$$

Note that $(\ell^\sigma_{ia})_{i\in I^\sigma,1\le a\le r}$ is the matrix inverse to $(m_{ia})_{i\in I^\sigma,1\le a\le r}$. Because $p_a\in\mathrm{cl}(\tilde{C}_\mathcal{X})\subset\sum_{i\in I^\sigma}\mathbb{R}_{\ge0}D_i$, it follows that $\ell^\sigma_{ia}\ge0$. It is also easy to see that $b^\sigma_{ij}$ is determined by $b_i=\sum_{j\notin I^\sigma}b^\sigma_{ij}b_j$ in $\boldsymbol{N}\otimes\mathbb{R}$. Let $V(\sigma)$ be $n!|\boldsymbol{N}_{\rm tor}|$ times the volume of the convex hull of $\{b_j\ ;\ j\notin I^\sigma\}\cup\{0\}$ in $\boldsymbol{N}\otimes\mathbb{R}$. Then the holomorphic volume form $\omega_q$ is given by

$$\omega_q=\frac{1}{V(\sigma)}\prod_{j\notin I^\sigma}\frac{dw_j}{w_j}.$$

We set

$$\mathbb{K}_{\mathrm{eff},\sigma}:=\{d\in\mathbb{L}\otimes\mathbb{Q}\ ;\ \langle D_i,d\rangle\in\mathbb{Z}_{\ge0},\forall i\in I^\sigma\}=\bigoplus_{i\in I^\sigma}\mathbb{Z}_{\ge0}\ell^\sigma_i.$$

Here, $\ell^\sigma_i\in\mathbb{L}\otimes\mathbb{Q}$ is defined by $\langle p_a,\ell^\sigma_i\rangle=\ell^\sigma_{ia}$. Then we have $\mathbb{K}_{\rm eff}=\bigcup_{\sigma\in\mathcal{X}^T}\mathbb{K}_{\mathrm{eff},\sigma}$. We denote by $\overline{p}_a(\sigma)$ and $\overline{D}_j(\sigma)$ the restrictions of $\overline{p}_a,\overline{D}_j\in H^*_T(\mathcal{X})$ to the fixed point $\sigma$. By using $\overline{D}_i(\sigma)=0$ for $i\in I^\sigma$ and (75), we calculate

$$\overline{p}_a(\sigma)=\sum_{i\in I^\sigma}\lambda_i\ell^\sigma_{ia},\quad\overline{D}_j(\sigma)=-\lambda_j-\sum_{i\in I^\sigma}\lambda_i b^\sigma_{ij},\quad j\notin I^\sigma. \tag{80}$$

For a function $f(q_1,\dots,q_r)$ in $(q_1,\dots,q_r)\in(\mathbb{R}_{>0})^r$, we write $f(q_1,\dots,q_r)=O(M)$ for $M\in\mathbb{R}$ when $f(tq_1,\dots,tq_r)=O(t^M)$ as $t\searrow+0$.

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[6] For the proof, see the revised version [45] of this paper.

**Lemma 4.22.** *Let $\sigma$ be a fixed point in $\mathcal{X}$. For any $M > 0$, there exists $M' > 0$ such that the following holds. For $\lambda_1, \dots, \lambda_m$ such that $\Re(-\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}}) > M'$ for all $j \notin I^\sigma$, $\mathcal{I}^\lambda_{\Gamma_0}(q, -1)$ with $(q_1, \dots, q_r) \in (\mathbb{R}_{>0})^r$ has the expansion*

$$\mathcal{I}^\lambda_{\Gamma_0}(q,-1) = \frac{(2\pi)^{n/2} e^{(\lambda_1+\cdots+\lambda_m)/2}}{\sqrt{-1}^n V(\sigma)} (e^{-\pi\sqrt{-1}\hat{\rho}}q)^{\frac{\overline{p}(\sigma)}{2\pi\sqrt{-1}}} \times$$

$$\left( \sum_{\substack{d\in\mathbb{K}_{\mathrm{eff},\sigma}, \\ |d|<M}} \frac{(e^{-\pi\sqrt{-1}\hat{\rho}}q)^d}{\prod_{j\notin I^\sigma}(1-e^{-2\pi\sqrt{-1}\langle D_j,d\rangle - \overline{D}_j(\sigma)}) \prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle + \frac{\overline{D}_i(\sigma)}{2\pi\sqrt{-1}})} + O(M) \right).$$

*where $|d| = \sum_{a=1}^r \langle p_a, d\rangle$ and we set*

$$(e^{-\pi\sqrt{-1}\hat{\rho}}q)^{\frac{\overline{p}(\sigma)}{2\pi\sqrt{-1}}} := \prod_{a=1}^r (e^{-\pi\sqrt{-1}\rho_a} q_a)^{\frac{\overline{p}_a(\sigma)}{2\pi\sqrt{-1}}}, \quad (e^{-\pi\sqrt{-1}\hat{\rho}}q)^d := \prod_{a=1}^r (e^{-\pi\sqrt{-1}\rho_a}q_a)^{\langle p_a, d\rangle}.$$

*Proof.* Using the notation above, we can write

$$\mathcal{I}^\lambda_{\Gamma_0}(q,-1) = \frac{q^{\frac{\overline{p}(\sigma)}{2\pi\sqrt{-1}}}}{(2\pi)^{n/2}V(\sigma)} \int_{(0,\infty)^n} \exp(-\sum_{i\in I^\sigma} q^{\ell_i^\sigma} w_\sigma^{b_i}) e^{-\sum_{j\notin I^\sigma} w_j} w_\sigma^{-\frac{\overline{D}(\sigma)}{2\pi\sqrt{-1}}} \frac{dw_\sigma}{w_\sigma}.$$

where we put $w_\sigma^{b_i} := \prod_{j\notin I^\sigma} w_j^{b_{ij}^\sigma}$, $w_\sigma^{-\frac{\overline{D}(\sigma)}{2\pi\sqrt{-1}}} := \prod_{j\notin I^\sigma} w_j^{-\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}}}$ and $dw_\sigma/w_\sigma := \prod_{j\notin I^\sigma}(dw_j/w_j)$. Consider the Taylor expansion:

$$\exp(-\sum_{i\in I^\sigma} q^{\ell_i^\sigma} w_\sigma^{b_i}) = \sum_{\substack{n_i\geq 0\ ;\ i\in I^\sigma, \\ |\sum_{i\in I^\sigma} n_i\ell_i^\sigma|<M}} \frac{\prod_{i\in I^\sigma}(-1)^{n_i} q^{n_i\ell_i^\sigma} w_\sigma^{n_i b_i}}{\prod_{i\in I^\sigma} n_i!} + O(M).$$

When $\Re(-\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}})$ is sufficiently big for all $j \notin I^\sigma$, each term in the right-hand side is integrable for the measure $e^{-\sum_{j\notin I^\sigma} w_j} w_\sigma^{-\frac{\overline{D}(\sigma)}{2\pi\sqrt{-1}}}(dw_\sigma/w_\sigma)$ on $(0,\infty)^n$. Therefore, we calculate

$$\mathcal{I}^\lambda_{\Gamma_0}(q,-1) = \frac{q^{\frac{\overline{p}(\sigma)}{2\pi\sqrt{-1}}}}{(2\pi)^{n/2}V(\sigma)} \left( \sum_{\substack{d\in\mathbb{K}_{\mathrm{eff},\sigma}, \\ |d|<M}} \frac{(-1)^{\sum_{i\in I^\sigma} n_i} q^d}{\prod_{i\in I^\sigma} n_i!} \prod_{j\notin I^\sigma} \Gamma\left( \sum_{i\in I^\sigma} n_i b_{ij}^\sigma - \frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}} \right) + O(M) \right),$$

where $d = \sum_{i\in I^\sigma} n_i \ell_i^\sigma$. Using $n_i = \langle D_i, d\rangle$, $\sum_{i\in I^\sigma} n_i b_{ij}^\sigma = -\langle D_j, d\rangle$ and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we arrive at the formula in the lemma. □

Next we study the asymptotic behavior of $H^\lambda_{\sigma,v}(q,-1)$ in the limit $q \searrow +0$. We can take the vector $\eta \in \mathbb{L}\otimes\mathbb{R}$ in our initial data of $\mathcal{X}$ to be $p_1 + \cdots + p_r \in \tilde{C}_{\mathcal{X}}$. Recall that $\mathcal{X}$ can be written as a symplectic quotient (55), so is endowed with a reduced symplectic form which depends on $\eta$. For simplicity, we assume that $\lambda_j$ is purely imaginary. Define a Hamiltonian function $\mathfrak{h}_{\eta,\lambda}\colon \mathcal{X} \to \mathbb{R}$ by

$$\mathfrak{h}_{\eta,\lambda}(z_1,\dots,z_m) = -\sum_{i=1}^m \frac{\lambda_j}{2\pi\sqrt{-1}} |z_j|^2, \quad (z_1,\dots,z_m) \in \mathfrak{h}^{-1}(\eta).$$

This generates a Hamiltonian $\mathbb{R}$-action $(z_1,\dots,z_m)\mapsto(e^{-\lambda_1 s}z_1,\dots,e^{-\lambda_m s}z_m)$, $s\in\mathbb{R}$ on $\mathcal{X}$. In general, the moment map for an $\mathbb{R}$-action preserving the complex structure on $\mathcal{X}$ attains its global maximum value at every critical point of index $2n=\dim_{\mathbb{R}}\mathcal{X}$. (This follows from the so-called Mountain-Path Lemma and the fact that there are no critical points of odd index. See *e.g.* [6]). Because the weights of $T_\sigma\mathcal{X}$ for this $\mathbb{R}$-action are $\{\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}}\ ;\ j\notin I^\sigma\}$, it follows that

$$\mathfrak{h}_{\eta,\lambda} \text{ attains its unique maximum value at } \sigma \iff -\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}}>0,\ \forall j\notin I^\sigma. \tag{81}$$

For a given $M>0$ and a fixed point $\sigma\in\mathcal{X}$, we can choose $\lambda_1,\dots,\lambda_m\in\sqrt{-1}\mathbb{R}$ such that the expansion in Lemma 4.22 holds. (This is possible since one can make $\Re(-\frac{\overline{D}_j(\sigma)}{2\pi\sqrt{-1}})$ arbitrarily large. See (80).) Then by (81), we know that $\mathfrak{h}_{\eta,\lambda}(\sigma)>\mathfrak{h}_{\eta,\lambda}(\sigma')$ for any other fixed point $\sigma'\neq\sigma$. On the other hand, we can easily see that $\mathfrak{h}_{\eta,\lambda}(\sigma)=-\sum_{a=1}^r\frac{\overline{p}_a(\sigma)}{2\pi\sqrt{-1}}$. Therefore, by rescaling $\lambda_i$ if necessary, we can assume that

$$\sum_{a=1}^r\frac{\overline{p}_a(\sigma)}{2\pi\sqrt{-1}}+M<\sum_{a=1}^r\frac{\overline{p}_a(\sigma')}{2\pi\sqrt{-1}},\quad \forall\sigma'\neq\sigma.$$

Then we have the following expansions:

$$\begin{aligned}H^\lambda_{\tau,v}(q,-1)&=\frac{(2\pi)^{n/2}e^{(\lambda_1+\cdots+\lambda_m)/2}}{\sqrt{-1}^n}(e^{-\pi\sqrt{-1}\hat{\rho}}q)^{\frac{\overline{p}(\sigma)}{2\pi\sqrt{-1}}}\\&\times\begin{cases}\displaystyle\sum_{\substack{d\in\mathbb{K}_{\mathrm{eff},\sigma};\\ \mathrm{inv}(v(d))=v,|d|<M}}\frac{(e^{-\pi\sqrt{-1}\hat{\rho}}q)^d}{\prod_{i=1}^m\Gamma(1+\langle D_i,d\rangle+\frac{\overline{D}_i(\sigma)}{2\pi\sqrt{-1}})}+O(M) & \text{if } \tau=\sigma;\\ O(M) & \text{if } \tau\neq\sigma.\end{cases}\end{aligned}$$

Comparing this with the expansion in Lemma 4.22, we conclude

$$c_{\sigma,v}(\lambda)=\frac{1}{V(\sigma)\prod_{i\notin I^\sigma}(1-e^{-2\pi\sqrt{-1}f_v([D_i])-\overline{D}_i(\sigma)})},$$

where $c_{\sigma,v}$ is the coefficient appearing in (79) and $f_v([D_i])\in[0,1)$ is the rational number associated to $[D_i]\in H^2(\mathcal{X},\mathbb{Z})$ (see Section 3 and (61)). Hence, we find

$$c_{\sigma,v}(\lambda)=\frac{1}{V(\sigma)}\frac{\mathrm{Td}^\lambda_{\mathcal{X}}|_{(\sigma,v)}}{e_T(T_\sigma\mathcal{X}_v)},$$

where $\mathrm{Td}^\lambda_{\mathcal{X}}|_{(\sigma,v)}$ is the restriction of the equivariant Todd class $\mathrm{Td}^\lambda_{\mathcal{X}}$ to the fixed point $(\sigma,v)$ in $I\mathcal{X}$ and $e_T(T_\sigma\mathcal{X}_v)$ is the $T$-equivariant Euler class, *i.e.* products of the $T$-weights of $T_\sigma\mathcal{X}_v$. Since $V(\sigma)$ is the order of the stabilizer at $\sigma\in\mathcal{X}$, the Equation (77) follows from the localization theorem in $T$-equivariant cohomology [5]. In the non-equivariant limit, we have (78). Because $[e^{W_q/z}\omega_q]$ and its derivatives $z\nabla_{a_1}\cdots z\nabla_{a_k}[e^{W_q/z}\omega_q]$ generate the $B$-model $\frac{\infty}{2}$VHS, it follows that $\Gamma_0$ corresponds to the linear form $(\cdot,\mathcal{O}_{\mathcal{X}})_{K(\mathcal{X})_{\mathbb{C}}}$.

4.4.4. *Proof of Theorem 4.17.* Let $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \subset \mathcal{V}^{\mathcal{X}}$ be the $\widehat{\Gamma}$-integral structure in Definition-Proposition 3.16 and $\tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}} \subset \mathcal{V}^{\mathcal{X}}$ be the integral structure pulled back from the B-model via mirror isomorphism (73). We know by Theorem 4.19 that the integral vector $\Gamma_0 \in \mathcal{R}^{\vee}_{\mathbb{Z}}$ corresponds to the linear form $\alpha \mapsto (\alpha, \Psi(\mathcal{O}_{\mathcal{X}}))_{\mathcal{V}^{\mathcal{X}}}$. Since the B-model integral structure is unimodular (and the B-model pairing corresponds to $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ by mirror conjecture assumption), it follows that $\Psi(\mathcal{O}_{\mathcal{X}}) \in \tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}}$. By Proposition 3.5, $\tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}}$ must be invariant under Galois action. Hence by Definition-Proposition 3.16, we know that $\Psi(\mathbb{Z}[\mathrm{Pic}(\mathcal{X})]\mathcal{O}_{\mathcal{X}}) \subset \tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}}$. Because $K(\mathcal{X})$ is generated by line bundles [10], we have $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \subset \tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}}$. Since the pairing $(\cdot,\cdot)_{\mathcal{V}^{\mathcal{X}}}$ is unimodular on $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}$ by assumption (A3), we must have $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} = \tilde{\mathcal{V}}^{\mathcal{X}}_{\mathbb{Z}}$.

## 5. Example: $tt^*$-geometry of $\mathbb{P}^1$

We calculate the Cecotti-Vafa structure on quantum cohomology of $\mathbb{P}^1$ with respect to the real structure induced from the $\widehat{\Gamma}$-integral structure in Definition-Proposition 3.16. By Theorem 4.17, this is the same as the Cecotti-Vafa structure associated to the Landau-Ginzburg model (mirror of $\mathbb{P}^1$):

$$\pi\colon (\mathbb{C}^*)^2 \to (\mathbb{C}^*), \quad (x,y) \mapsto q = xy, \quad W = x + y.$$

Let $\omega \in H^2(\mathbb{P}^1)$ be the unique integral Kähler class. Let $\{t^0, t^1\}$ be the linear coordinate system on $H^*(\mathbb{P}^1)$ dual to the basis $\{\mathbf{1}, \omega\}$. Put $\tau = t^0\,\mathbf{1} + t^1\omega$. The quantum product $\circ_\tau$ is given by

$$(\mathbf{1}\circ_\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\omega\circ_\tau) = \begin{bmatrix} 0 & e^{t^1} \\ 1 & 0 \end{bmatrix},$$

where we identify $\mathbf{1}, \omega$ with column vectors $[1,0]^{\mathrm{T}}$, $[0,1]^{\mathrm{T}}$ and the matrices act on vectors by the left multiplication. The exponential $e^{t^1}$ corresponds to $q$ in the Landau-Ginzburg model via the mirror map, so we set $q = e^{t^1}$. Hereafter, we restrict $\tau$ to lie on $H^2(\mathbb{P}^1)$ but we will not lose any information by this (see Remark 5.1 below). Recall that the Hodge structure $\mathbb{F}_\tau$ associated with the quantum cohomology of $\mathbb{P}^1$ is given by the image of $\mathcal{J}_\tau\colon H^*(\mathbb{P}^1)\otimes\mathbb{C}\{z\} \to \mathcal{H}^{\mathbb{P}^1} = H^*(\mathbb{P}^1)\otimes\mathbb{C}\{z,z^{-1}\}$ in (28). The map $\mathcal{J}_\tau$ is given by the explicit hypergeometric function $J(q,z)$ [35]:

$$\mathcal{J}_\tau = \begin{pmatrix} | & | \\ J(e^{t^1}, z) & z\partial_1 J(e^{t^1}, z) \\ | & | \end{pmatrix} = e^{t^1\omega/z}\circ Q, \quad Q := \begin{pmatrix} J_0 & z\partial_1 J_0 \\ J_1/z & J_0 + \partial_1 J_1 \end{pmatrix},$$

$$J(q,z) := e^{t^1\omega/z}\sum_{k=0}^{\infty}\frac{q^k\,\mathbf{1}}{(\omega+z)^2\cdots(\omega+kz)^2} = e^{t^1\omega/z}(J_0(q,z)\,\mathbf{1} + J_1(q,z)\frac{\omega}{z}),$$

where $\partial_1 = (\partial/\partial t^1)$. By Definition-Proposition 3.16, an integral basis of $\mathcal{V}^{\mathbb{P}^1} = H^*(\mathbb{P}^1)$ is given by

$$\Psi(\mathcal{O}_{\mathbb{P}^1}) = \frac{1}{\sqrt{2\pi}}(\mathbf{1} - 2\gamma\omega), \quad \Psi(\mathcal{O}_{\mathrm{pt}}) = \sqrt{2\pi}\sqrt{-1}\omega,$$

where $\gamma$ is the Euler constant. Hence the real involutions on $\mathcal{V}^{\mathbb{P}^1}$ and $\mathcal{H}^{\mathbb{P}^1}$ are given respectively by (see (39)):

$$\kappa_{\mathcal{V}} = \begin{bmatrix} 1 & 0 \\ -4\gamma & -1 \end{bmatrix} \circ \overline{\phantom{x}}, \quad \kappa_{\mathcal{H}} = \begin{bmatrix} z & 0 \\ -4\gamma & -z^{-1} \end{bmatrix} \circ \overline{\phantom{x}}.$$

where $\overline{\phantom{x}}$ is the usual complex conjugation (when $z$ is on $S^1 = \{|z| = 1\}$).

To obtain the Cecotti-Vafa structure, we need to find a basis of $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$. The procedure below follows the proof of Theorem 3.7 in Section 3.4. Put $\mathbb{F}'_\tau := e^{-t^1\omega/z}\mathbb{F}_\tau$ and $\kappa_{\mathcal{H}}^\tau := e^{-(t^1+\overline{t^1})\omega/z}\kappa_{\mathcal{H}}$. By

$$\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) = e^{t^1\omega/z}(\mathbb{F}'_\tau \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}'_\tau)),$$

it suffices to calculate a basis of $\mathbb{F}'_\tau \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}'_\tau)$. First we approximate $\mathbb{F}'_\tau$ by $\mathbb{F}_{\lim} := H^*(\mathbb{P}^1) \otimes \mathbb{C}\{z\}$ and solve for a basis of $\mathbb{F}_{\lim} \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}_{\lim})$. By elementary linear algebra, we find the following Birkhoff factorization of $[\kappa_{\mathcal{H}}^\tau(\mathbf{1}), \kappa_{\mathcal{H}}^\tau(\omega)]$:

$$[\kappa_{\mathcal{H}}^\tau(\mathbf{1}), \kappa_{\mathcal{H}}^\tau(\omega)] = BC, \quad B := \begin{bmatrix} 1 & z/a_\tau \\ 0 & 1 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 1/a_\tau \\ a_\tau & -1/z \end{bmatrix},$$

where $a_\tau := -t^1 - \overline{t^1} - 4\gamma$. Then the column vectors of $B$ give a basis of $\mathbb{F}_{\lim} \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}_{\lim})$ (*c.f.* (13)). Note that the column vectors of $Q$ above form a basis of $\mathbb{F}'_\tau$. Thus it suffices to calculate the Birkhoff factorization of $Q^{-1}\kappa_{\mathcal{H}}^\tau(Q)$ to solve for a basis of $\mathbb{F}'_\tau \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}'_\tau)$. Define a matrix $S$ by

$$\kappa_{\mathcal{H}}^\tau(Q) = QBSC.$$

Using that $Q^{-1}$ is given by the adjoint of $Q(-z)$ (by Proposition 3.3), we have

$$S = \begin{bmatrix} \begin{matrix} 2\Re(J_0\overline{J_1})a_\tau^{-1} + |J_0|^2 + 2\Re(\partial_1 J_0 \overline{J_1} + J_0 \overline{\partial_1 J_1}) \\ +2\Re(\partial_1 J_0 \overline{\partial_1 J_1})a_\tau - |\partial_1 J_0|^2 a_\tau^2 \end{matrix} & \begin{matrix} (2\Re(J_0\overline{J_1})a_\tau^{-2} \\ +(\partial_1 J_0 \overline{J_1} + \overline{J_0}\partial_1 J_1)a_\tau^{-1} - \partial_1 J_0 \overline{J_0})z \end{matrix} \\ \begin{matrix} (-2\Re(J_0\overline{J_1}) - (\overline{\partial_1 J_0 J_1 + J_0 \partial_1 J_1})a_\tau \\ + J_0 \overline{\partial_1 J_0} a_\tau^2)z^{-1} \end{matrix} & -2\Re(J_1\overline{J_0})a_\tau^{-1} + |J_0|^2 \end{bmatrix},$$

where we restrict $z$ to lie on $S^1 = \{|z| = 1\}$. Because $S = \mathbf{1} + O(|q|^{1-\epsilon})$, $\epsilon > 0$ as $|q| \to 0$, this admits the Birkhoff factorization $S = \tilde{B}\tilde{C}$ for $|q| \ll 1$, where $\tilde{B} \colon \mathbb{D}_0 \to GL_2(\mathbb{C})$, $\tilde{C} \colon \mathbb{D}_\infty \to GL_2(\mathbb{C})$ such that $\tilde{B}(0) = \mathbf{1}$. Then the column vectors of $QB\tilde{B} = \kappa_{\mathcal{H}}^\tau(Q)C^{-1}\tilde{C}^{-1}$ give a basis of $\mathbb{F}'_\tau \cap \kappa_{\mathcal{H}}^\tau(\mathbb{F}'_\tau)$. We perform the Birkhoff factorization in the following way. Note that $S$ is expanded in a power series in $q$ and $\overline{q}$ with coefficients in Laurent polynomials in $a_\tau$ and $z$:

$$S = \sum_{n,m\ge 0} S_{n,m} q^n \overline{q}^m, \quad S_{n,m} \in \mathrm{End}(\mathbb{C}^2)[z, z^{-1}, a_\tau, a_\tau^{-1}].$$

We put $\tilde{B} = \sum_{n,m\ge 0} \tilde{B}_{n,m} q^n \overline{q}^m$, $\tilde{C} = \sum_{n,m\ge 0} \tilde{C}_{n,m} q^n \overline{q}^m$. Since $S_{0,0} = \tilde{B}_{0,0} = \tilde{C}_{0,0} = \mathrm{id}$, we can recursively solve for $\tilde{B}_{n,m}$ and $\tilde{C}_{n,m}$ by decomposing

$$\tilde{B}_{n,m} + \tilde{C}_{n,m} = S_{n,m} - \sum_{(i,j)\ne 0, (n-i,m-j)\ne 0} \tilde{B}_{i,j}\tilde{C}_{n-i,m-j}$$

into strictly positive power series $\tilde{B}_{n,m}$ and non-positive power series $\tilde{C}_{n,m}$ in $z$. The first six terms of $B\tilde{B}$ are given by

$$
\begin{aligned}
B\tilde{B} = & \begin{bmatrix} 1 & \frac{z}{a_\tau} \\ 0 & 1 \end{bmatrix} + \overline{q} \begin{bmatrix} (1+a_\tau)z^2 & \frac{z^3}{a_\tau} \\ (2+2a_\tau+a_\tau^2)z & \frac{(2+a_\tau)z^2}{a_\tau} \end{bmatrix} + q\overline{q} \begin{bmatrix} 0 & -\frac{(8+8a_\tau+2a_\tau^2)z}{a_\tau^2} \\ 0 & 0 \end{bmatrix} \\
& + \overline{q}^2 \begin{bmatrix} \frac{(1+2a_\tau)z^4}{4} & \frac{z^5}{4a_\tau} \\ \frac{(3+6a_\tau+2a_\tau^2)z^3}{4} & \frac{(3+a_\tau)z^4}{4a_\tau} \end{bmatrix} + q\overline{q}^2 \begin{bmatrix} \frac{(33+34a_\tau+18a_\tau^2+4a_\tau^3)z^2}{4} & -\frac{(32+31a_\tau+12a_\tau^2+2a_\tau^3)z^3}{4a_\tau^2} \\ \frac{(25+50a_\tau+34a_\tau^2+12a_\tau^3+2a_\tau^4)z}{2} & -\frac{(64+78a_\tau+45a_\tau^2+14a_\tau^3+2a_\tau^4)z^2}{4a_\tau^2} \end{bmatrix} \\
& + \overline{q}^3 \begin{bmatrix} \frac{(1+3a_\tau)z^6}{36} & \frac{z^7}{36a_\tau} \\ \frac{(11+33a_\tau+9a_\tau^2)z^5}{108} & \frac{(11+3a_\tau)z^6}{108a_\tau} \end{bmatrix} + O((\log|q|)^5|q|^4)
\end{aligned}
$$

Let $\Phi_\tau$ denote the inverse to the natural projection $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau) \to \mathbb{F}_\tau/z\mathbb{F}_\tau = H^*(\mathbb{P}^1)$. Because $B\tilde{B} = \mathbf{1} + O(z)$, we have $[\Phi_\tau(\mathbf{1}), \Phi_\tau(\omega)] = e^{t^1\omega/z}QB\tilde{B}$:

$$
\Phi_\tau \colon H^*(\mathbb{P}^1) = \mathbb{F}'_\tau/z\mathbb{F}'_\tau \xrightarrow{QB\tilde{B}} \mathbb{F}'_\tau \cap \kappa^\tau_{\mathcal{H}}(\mathbb{F}'_\tau) \xrightarrow{e^{t^1\omega/z}} \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau).
$$

The Cecotti-Vafa structure for $\mathbb{P}^1$ is defined on the trivial vector bundle $K := H^*(\mathbb{P}^1) \times H^*(\mathbb{P}^1) \to H^*(\mathbb{P}^1)$. Recall that the Hermitian metric $h$ on $K_\tau$ is the pull-back of the Hermitian metric $(\alpha, \beta) \mapsto (\kappa_{\mathcal{H}}(\alpha), \beta)_{\mathcal{H}}$ on $\mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$ through $\Phi_\tau \colon K_\tau \cong \mathbb{F}_\tau \cap \kappa_{\mathcal{H}}(\mathbb{F}_\tau)$. The Hermitian metric $h$ is of the form:

$$
h = \begin{bmatrix} h_{\overline{0}0} & 0 \\ 0 & h_{\overline{0}0}^{-1} \end{bmatrix}, \quad h_{\overline{0}0} := \int_{\mathbb{P}^1} \kappa_{\mathcal{H}}(\Phi_\tau(\mathbf{1}))\Big|_{z\mapsto -z} \cup \Phi_\tau(\mathbf{1}).
$$

The first seven terms of the expansion of $h_{\overline{0}0}$ are (with $a_\tau = -t^1 - \overline{t^1} - 4\gamma$, $q = e^{t^1}$)

$$
\begin{aligned}
h_{\overline{0}0} = & a_\tau + |q|^2\left(a_\tau^3 + 4a_\tau^2 + 8a_\tau + 8\right) + |q|^4\left(a_\tau^5 + 8a_\tau^4 + \frac{121}{4}a_\tau^3 + \frac{129}{2}a_\tau^2 + \frac{145}{2}a_\tau + \frac{145}{4}\right) \\
& + |q|^6\left(a_\tau^7 + 12a_\tau^6 + \frac{275}{4}a_\tau^5 + \frac{477}{2}a_\tau^4 + \frac{9539}{18}a_\tau^3 + \frac{81001}{108}a_\tau^2 + \frac{50342}{81}a_\tau + \frac{55526}{243}\right) \\
& + |q|^8\left(a_\tau^9 + 16a_\tau^8 + \frac{493}{4}a_\tau^7 + \frac{1185}{2}a_\tau^6 + \frac{31001}{16}a_\tau^5 + \frac{79939}{18}a_\tau^4 + \frac{49077907}{6912}a_\tau^3\right. \\
& \qquad \left. + \frac{52563371}{6912}a_\tau^2 + \frac{614694323}{124416}a_\tau + \frac{736622003}{497664}\right) \\
& + |q|^{10}\left(a_\tau^{11} + 20a_\tau^{10} + \frac{775}{4}a_\tau^9 + \frac{2381}{2}a_\tau^8 + \frac{368599}{72}a_\tau^7 + \frac{1738481}{108}a_\tau^6 + \frac{780126811}{20736}a_\tau^5\right. \\
& \qquad \left. + \frac{4053627445}{62208}a_\tau^4 + \frac{254355946241}{3110400}a_\tau^3 + \frac{1465574917127}{20736000}a_\tau^2 + \frac{163291639271}{4320000}a_\tau + \frac{1840366543439}{194400000}\right) \\
& + |q|^{12}\left(a_\tau^{13} + 24a_\tau^{12} + \frac{1121}{4}a_\tau^{11} + \frac{4193}{2}a_\tau^{10} + \frac{1606399}{144}a_\tau^9 + \frac{2398517}{54}a_\tau^8 + \frac{2814667745}{20736}a_\tau^7 + \frac{20004983519}{62208}a_\tau^6\right. \\
& \qquad + \frac{407437321759}{691200}a_\tau^5 + \frac{51278023471273}{62208000}a_\tau^4 + \frac{796478452045403}{933120000}a_\tau^3 + \frac{11553263487112967}{18662400000}a_\tau^2 \\
& \qquad \left. + \frac{11823418405646927}{41990400000}a_\tau + \frac{15268380040196927}{251942400000}\right) + \cdots.
\end{aligned}
$$

The other data $(\kappa, g, C, \tilde{C}, D, \mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q})$ of the Cecotti-Vafa structure are given in terms of $h_{\overline{0}0}$. In fact, we have $C_0 = \widetilde{C}_{\overline{0}} = \mathrm{id}$, $D_0 = \partial/\partial t^0$, $D_{\overline{0}} = \partial/\partial \overline{t^0}$ and

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 0 & h_{\overline{0}0}^{-1} \\ h_{\overline{0}0} & 0 \end{bmatrix} \circ \overline{\phantom{x}}, \quad D_1 = \partial_1 + \begin{bmatrix} \partial_1 \log h_{\overline{0}0} & 0 \\ 0 & -\partial_1 \log h_{\overline{0}0} \end{bmatrix},$$

$$D_{\overline{1}} = \overline{\partial_1}, \quad C_1 = \frac{1}{2}\mathcal{U} = \begin{bmatrix} 0 & e^{t^1} \\ 1 & 0 \end{bmatrix}, \quad \widetilde{C}_{\overline{1}} = \frac{1}{2}\overline{\mathcal{U}} = \begin{bmatrix} 0 & h_{\overline{0}0}^{-2} \\ e^{\overline{t^1}} h_{\overline{0}0}^2 & 0 \end{bmatrix},$$

$$\mathcal{Q} = \partial_E + \mu - D_E = \begin{bmatrix} -\frac{1}{2} - 2\partial_1 \log h_{\overline{0}0} & 0 \\ 0 & \frac{1}{2} + 2\partial_1 \log h_{\overline{0}0} \end{bmatrix},$$

where $\partial, \overline{\partial}$ are the connections given by the given trivialization of $K$.

**Remark 5.1.** (i) The fact that the Hermitian metric $h$ is represented by a diagonal matrix with determinant 1 follows from an elementary argument. See [66, Lemma 2.1].

(ii) From the general theory of (trTERP)+(trTLEP) structure on the tangent bundle, it follows that $h, C, \widetilde{C}, \mathcal{U}, \overline{\mathcal{U}}, \mathcal{Q}$ are invariant under the flow of the unit vector field $(\partial/\partial t^0), (\partial/\partial \overline{t^0})$. Therefore, the calculation here determines the Cecotti-Vafa structure on the big quantum cohomology. Moreover we have $D_E + \mathcal{Q} = \partial_E + \mu$ and $\mathrm{Lie}_{E - \overline{E}} h = 0$. In the case of $\mathbb{P}^1$, this means that $h_{\overline{0}0}$ depends only on $|q|$. See [38].

(iii) We can show that our procedure for the Birkhoff factorization gives convergent series for sufficiently small values of $|q|$. In particular, the expansion for $h_{\overline{0}0}$ converges for small $|q|$.

Our calculation of the Hermitian metric $h_{\overline{0}0}$ matches with Cecotti-Vafa's result [16] for the sigma model of $\mathbb{P}^1$. The $tt^*$-equation $[D_1, D_{\overline{1}}] + [C_1, \widetilde{C}_{\overline{1}}] = 0$ gives the following differential equation for $h_{\overline{0}0}$:

$$\partial_1 \overline{\partial_1} \log h_{\overline{0}0} = -h_{\overline{0}0}^{-2} + |q|^2 h_{\overline{0}0}^2. \tag{82}$$

In [16], $h_{\overline{0}0}$ was identified with a unique solution to (82) expanded in the form

$$h_{\overline{0}0} = \sum_{n=0}^{\infty} F_n |q|^{2n}, \quad F_0 = a_\tau, \quad F_n \in \mathbb{C}[a_\tau, a_\tau^{-1}], \quad a_\tau = -2\log|q| - 4\gamma.$$

The equation (82) gives an infinite set of recursive differential equations for $F_n$. It is easy to check that the differential equations determine the Laurent polynomial $F_n$ *uniquely*; Moreover it turns out that $F_n \in \mathbb{Q}[a_\tau]$ and $\deg F_n = 2n+1$. The *existence* of such a solution seems to be non-trivial, however our Birkhoff factorization method certainly gives such $h_{\overline{0}0}$. The differential equation (82) is equivalent to Painlevé III equation [15]:

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} = 4\sinh(u), \quad h_{\overline{0}0} = e^{u/2}|e^{-t^1/2}|, \quad z = 4|e^{t^1/2}|.$$

that $h_{\overline{0}0}$ here is positive and regular on the positive real axis $|q| \in (0, \infty)$[7]. By physical arguments, Cecotti-Vafa [15, 18, 16] showed that $h_{\overline{0}0}$ is positive and smooth on the positive real axis $0 < |q| < \infty$. Since the Landau-Ginzburg mirror of $\mathbb{P}^1$ is defined by a cohomologically tame function, this fact also follows from Sabbah's result [63] in the

[7] For this, the constant $\gamma$ in $a_\tau$ must be the very Euler constant.

singularity theory (see Remark 3.10). Therefore, the Cecotti-Vafa structure for $\mathbb{P}^1$ is well-defined and positive definite on the whole $H^*(\mathbb{P}^1)$. It seems that the same solution as $h_{\overline{0}0}$ has been obtained in the study of Painlevé III equation [46, 52] (the first few terms of the expansion are the same as ours). If this is the case, $h_{\overline{0}0}$ should have the asymptotics [46, 52] (also appearing in [16]):

$$h_{\overline{0}0} \sim \frac{1}{\sqrt{|q|}}\left(1 - \frac{1}{2\sqrt{\pi}|q|^{1/4}} e^{-8|q|^{1/2}}\right)$$

as $|q| \to \infty$. With respect to the metric $h_{\overline{1}1} = h_{\overline{0}0}^{-1}$ on the Kähler moduli space $H^2(\mathbb{P}^1)/2\pi\sqrt{-1}H^2(\mathbb{P}^1,\mathbb{Z})$, a neighborhood of the large radius limit point $q = 0$ has negative curvature, but does not have finite volume. The curvature $-\frac{2}{h_{\overline{0}0}}(1 - |q|^2 h_{\overline{0}0}^4)$ goes to zero as $|q| \to 0$ and $|q| \to \infty$ and the total curvature is $-\pi/4$. Much more examples including $\mathbb{P}^n$, $\mathbb{P}^1/\mathbb{Z}_n$ are calculated in physics literature. We refer the reader to [15, 16, 17].

## 6. Integral periods and Ruan's conjecture

In mirror symmetry for Calabi-Yau manifolds (see *e.g.* [14, 54, 28]), flat co-ordinates (or mirror map) $\tau_i$ on the B-model in a neighborhood of a maximally unipotent monodromy point was given by periods over integral cycles $A_1, \dots, A_r$ of a holomorphic $n$-form $\Omega$

$$\tau_i = \int_{A_i} \Omega,$$

where $\Omega$ is normalized by the condition:

$$\int_{A_0} \Omega = 1.$$

Here, $A_0$ is a monodromy-invariant cycle (unique up to sign) and $A_1, \dots, A_r$ are such that $\mathbb{Z}A_0 \oplus \sum_{i=1}^r \mathbb{Z}A_i$ is preserved under monodromy transformations. We should note that *flat co-ordinates are constructed as integral periods.* In this section, by choosing an integral structure on the A-model, we define integral periods of quantum cohomology analogously. We study relationships between integral periods and flat co-ordinates in the conformal limit (84). Using integral periods, we will speculate on specialization values of quantum parameters appearing in Ruan's crepant resolution conjecture. From this viewpoint, the specialization to roots of unity seems to be natural. Throughout this section, we assume that $\mathcal{X}$ is a weak Fano (*i.e.* $\rho = c_1(\mathcal{X})$ is nef) Gorenstein orbifold without generic stabilizer.

6.1. **Integral periods.** The integral periods for a general $\frac{\infty}{2}$VHS can be defined in the following way. Recall that a choice of integral structures defines an integral lattice $\mathcal{V}_\mathbb{Z}$ in the space $\mathcal{V}$ of multi-valued flat sections of $(\mathsf{H}, \widehat{\nabla}_{z\partial z})$ on $\mathbb{C}^*$. Take a basis $\Gamma_1, \dots, \Gamma_N$ of $\mathcal{V}_\mathbb{Z}^\vee$. Each $\Gamma_i$ defines a multi-valued section $\Gamma_i(\log z)$ of the dual bundle $\mathcal{H}^\vee \to \mathbb{C}^*$. Take a $\mathbb{C}\{z\}$-basis $s_1(\tau, z), \dots, s_N(\tau, z)$ of the Hodge structure $\mathbb{F}_\tau \subset \mathcal{H}$. We call the pairing

$$\tau \mapsto \langle \Gamma_i(\log z), s_j(\tau, z)\rangle$$

an *integral period*. For the Landau-Ginzburg mirror of toric orbifolds, these integral periods are given by oscillatory integrals over Lefschetz thimbles. These periods are themselves multi-valued functions in $z$ and not easy to understand. They will become more tractable if we choose $\Gamma_i$ to be invariant under monodromy transformations around $z = \infty$.

6.2. **A-model integral periods in the conformal limit.** Consider the A-model $\frac{\infty}{2}$VHS of $\mathcal{X}$ with an integral structure. The assumption that $\mathcal{X}$ is Gorenstein implies that the age $\iota_v$ is an integer for all $v \in \mathsf{T}$. Therefore, $H^*_{\rm orb}(\mathcal{X})$ is graded by even integers. Thus by (35), the monodromy transformation $M_z \in \operatorname{End}(\mathcal{V}^{\mathcal{X}})$ around $z = \infty$ is of the form:

$$M_z = (-1)^n e^{2\pi\sqrt{-1}\rho}$$

where $n = \dim_{\mathbb{C}} \mathcal{X}$ and $\rho = c_1(\mathcal{X})$. We define $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} \subset H^*_{\rm orb}(\mathcal{X})$ by

$$\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} := \operatorname{Ker}(\operatorname{id} - M_z^2) \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} = \operatorname{Ker}(\rho) \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z}}.$$

Under the map (32), an element of $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ corresponds to a flat section of $(\mathsf{H}^{\mathcal{X}}, \widehat{\nabla}_{z\partial_z})$ which is single-valued (when $n$ is even) or two-valued (when $n$ is odd). For convenience, we introduce the space $\widehat{\mathcal{H}}^{\mathcal{X}}$ of possibly two-valued sections of $\mathsf{H}^{\mathcal{X}} \to \mathbb{C}^*$:

$$\widehat{\mathcal{H}}^{\mathcal{X}} = \mathcal{H}^{\mathcal{X}} \otimes_{\mathbb{C}\{z,z^{-1}\}} \mathbb{C}\{z^{1/2}, z^{-1/2}\}.$$

The pairing on $\mathcal{H}^{\mathcal{X}}$ is extended on $\widehat{\mathcal{H}}^{\mathcal{X}}$ by

$$(\alpha, \beta)_{\mathcal{H}^{\mathcal{X}}} = (\alpha(\sqrt{-1}z^{1/2}), \beta(z^{1/2}))_{\rm orb}$$

where we regard $\alpha, \beta \in \widehat{\mathcal{H}}^{\mathcal{X}}$ as cohomology-valued functions in $z^{1/2}$. Under (32), $A \in \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ corresponds to $z^{-\mu}A \in \widehat{\mathcal{H}}^{\mathcal{X}}$ and gives an integral period:

$$\tau \mapsto (z^{-\mu}A, \mathcal{J}_\tau(\alpha))_{\mathcal{H}^{\mathcal{X}}} \in \mathbb{C}\{z^{1/2}, z^{-1/2}\}, \tag{83}$$

where $\alpha \in H^*_{\rm orb}(\mathcal{X}) \otimes \mathbb{C}\{z\}$ and $\mathcal{J}_\tau \colon H^*_{\rm orb}(\mathcal{X}) \otimes \mathbb{C}\{z\} \to \mathcal{H}^{\mathcal{X}}$ is an embedding in (28). These integral periods behaves well in the following limit:

$$\tau - s\rho, \quad \Re(s) \to \infty \tag{84}$$

with a fixed $\tau \in H^2_{\rm orb}(\mathcal{X})$. We call such a sequence in $H^2_{\rm orb}(\mathcal{X})$ the *conformal limit*. Recall that we assumed that $\rho = c_1(\mathcal{X})$ is nef. For the embedding $\mathcal{J}_\tau$ in (28), we define $\mathcal{J}^{\rm CY}_\tau \colon H^*_{\rm orb}(\mathcal{X}) \otimes \mathbb{C}\{z\} \to \mathcal{H}^{\mathcal{X}}$ as

(85)

$$\begin{aligned} \mathcal{J}^{\rm CY}_\tau(\alpha) &:= \lim_{\Re(s)\to\infty} e^{s\rho/z} \mathcal{J}_{\tau - s\rho}(\alpha) \\ &= e^{\tau_{0,2}/z}\left(\alpha + \sum_{\substack{(d,l)\neq(0,0),\\ d\in\operatorname{Ker}(\rho)}} \sum_{i=1}^{N} \frac{1}{l!}\left\langle \alpha, \tau_{\rm tw}, \dots, \tau_{\rm tw}, \frac{\phi_i}{z-\psi}\right\rangle^{\mathcal{X}}_{0,l+2,d} e^{\langle \tau_{0,2}, d\rangle} \phi^i \right). \end{aligned}$$

Here we put $\tau = \tau_{0,2} + \tau_{\rm tw}$ with $\tau_{0,2} \in H^2(\mathcal{X})$ and $\tau_{\rm tw} \in \bigoplus_{\iota_v=1} H^0(\mathcal{X}_v)$ and used that $\langle \rho, d\rangle \geq 0$ for all $d \in \operatorname{Eff}_{\mathcal{X}}$. When $\alpha \in H^{2k}_{\rm orb}(\mathcal{X})$, $\mathcal{J}^{\rm CY}_\tau(\alpha)$ is homogeneous of degree $2k$ if we set $\deg(z) = 2$. From this calculation, the following definition makes sense.

**Definition 6.1.** Assume that $\rho = c_1(\mathcal{X})$ is nef. Then we can define a new $\frac{\infty}{2}$VHS $\mathbb{F}^{\rm CY}_\tau \subset \mathcal{H}^{\mathcal{X}}$ (in the moving subspace realization) by

$$\mathbb{F}^{\rm CY}_\tau := \lim_{\Re(s)\to\infty} e^{s\rho/z}\mathbb{F}_{\tau-s\rho} = \mathcal{J}^{\rm CY}_\tau(H^*_{\rm orb}(\mathcal{X})\otimes\mathbb{C}\{z\}), \quad \tau\in H^2_{\rm orb}(\mathcal{X}).$$

This satisfies $\mathbb{F}^{\rm CY}_{\tau+a\rho} = e^{a\rho/z}\mathbb{F}^{\rm CY}_\tau$ and is homogeneous $(z\partial_z+\mu)\mathbb{F}^{\rm CY}_\tau \subset \mathbb{F}^{\rm CY}_\tau$.

**Remark 6.2.** We can also define the new $\frac{\infty}{2}$VHS above as a sheaf of $\mathcal{O}_{H^2_{\rm orb}(\mathcal{X})}\{z\}$-modules, using Dubrovin connection associated with a new quantum product $\circ^{\rm CY}_\tau := \lim_{\Re(s)\to\infty}\circ_{\tau-s\rho}$. The conformal limit of quantum cohomology is closely related to Y. Ruan's quantum corrected ring [61], which is defined by counting rational curves contained in the exceptional locus (in the case of crepant resolution). The conformal limit of a $\frac{\infty}{2}$VHS appears in the work of Sabbah [62, Part I] as the associated graded of a free $\mathbb{C}[z]$-module $G_k$ (an algebraization of $z^{-k}\mathbb{F}_\tau$) with respect to the Kashiwara-Malgrange $V$-filtration at $z=\infty$. See also Hertling and Sevenheck [39, Section 7] for a review.

Because of the homogeneity of $\mathbb{F}^{\rm CY}_\tau$, the $\frac{\infty}{2}$VHS $\{\mathbb{F}^{\rm CY}_\tau \subset \mathcal{H}^{\mathcal{X}}\}$ reduces to a *finite dimensional* VHS. Set $\hat{\mathbb{F}}^{\rm CY}_\tau := \mathbb{F}^{\rm CY}_\tau \otimes_{\mathbb{C}\{z\}}\mathbb{C}\{z^{1/2}\} \subset \widehat{\mathcal{H}}^{\mathcal{X}}$ and $H_0 := \operatorname{Ker}(z\partial_z+\mu) \subset \widehat{\mathcal{H}}^{\mathcal{X}}$. By restriction, the pairing on $\widehat{\mathcal{H}}^{\mathcal{X}}$ induces a $(-1)^n$-symmetric $\mathbb{C}$-valued pairing $(\cdot,\cdot)_{H_0}$ on $H_0$. By restricting the semi-infinite flag $\cdots\supset z^{-1}\hat{\mathbb{F}}^{\rm CY}_\tau \supset \hat{\mathbb{F}}^{\rm CY}_\tau \supset z\hat{\mathbb{F}}^{\rm CY}_\tau \supset\cdots$ to $H_0$, we obtain a finite dimensional flag $H_0 = F^0_\tau \supset F^1_\tau \supset \cdots \supset F^n_\tau \supset 0$:

$$\begin{aligned} F^p_\tau &:= z^{p-n/2}\hat{\mathbb{F}}^{\rm CY}_\tau \cap H_0 \\ &= \operatorname{Span}\left\{ z^{p-n/2}\mathcal{J}^{\rm CY}_\tau(z^j\alpha)\ ;\ \alpha\in H^{2n-2p-2j}_{\rm orb}(\mathcal{X}), j\ge 0\right\}. \end{aligned}$$

One can check that $F^p_\tau$ satisfies the Griffiths transversality and Hodge-Riemann bilinear relation:

$$\frac{\partial}{\partial t^i}F^p_\tau \subset F^{p-1}_\tau, \quad (F^p_\tau, F^{n-p+1}_\tau)_{H_0} = 0.$$

The real involution $\kappa_{\mathcal{H}}$ on $\mathcal{H}^{\mathcal{X}}$ induces those on $\widehat{\mathcal{H}}^{\mathcal{X}}$ and $H_0$ since $z\partial_z+\mu$ is purely imaginary on $\mathcal{H}^{\mathcal{X}}$ by (38). Denote by $\kappa_{H_0}$ the real involution on $H_0$. When moreover $\mathbb{F}^{\rm CY}_\tau$ is pure and polarized (these properties hold near the large radius limit if the conditions of Theorem 3.7 are satisfied), one can easily check that $F^p_\tau$ satisfies the Hodge decomposition and Hodge-Riemann bilinear inequality:

$$H_0 = F^p_\tau \oplus \kappa_{H_0}(F^{n-p+1}_\tau), \quad (-\sqrt{-1})^{2p-n}(\phi,\kappa_{H_0}(\phi))_{H_0} > 0$$

where $\phi \in F^p_\tau\cap\kappa_{H_0}(F^{n-p}_\tau) = z^{p-n/2}(\hat{\mathbb{F}}^{\rm CY}_\tau \cap \kappa_{\mathcal{H}}(\hat{\mathbb{F}}^{\rm CY}_\tau))\cap H_0$. Conversely, this finite dimensional VHS $F^\bullet_\tau$ recovers the $\frac{\infty}{2}$VHS $\mathbb{F}^{\rm CY}_\tau$ by

$$\mathbb{F}^{\rm CY}_\tau = z^{-n/2}F^n_\tau\otimes\mathbb{C}\{z\} + z^{-n/2+1}F^{n-1}_\tau\otimes\mathbb{C}\{z\} + \cdots + z^{n/2}F^0_\tau\otimes\mathbb{C}\{z\}.$$

In contrast to the real structure, the integral structure on the A-model $\frac{\infty}{2}$VHS does not induce a full integral lattice of $H_0$. One can see however that the lattice $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ is naturally contained in $H_0$ by $A\mapsto z^{-\mu}A$ as a *partial* lattice. The pairing between $z^{-\mu}A$

with $A \in \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ and a section of $F^p_\tau$ gives an integral period for $F^p_\tau$. Let us consider the conformal limit of the A-model integral period (83) with $\alpha \in H^{2p}_{\rm orb}(\mathcal{X})$:

$$\begin{aligned}\lim_{\Re(s)\to\infty} (z^{-\mu}A, \mathcal{J}_{\tau-s\rho}(\alpha))_{\mathcal{H}^{\mathcal{X}}} &= \lim_{\Re(s)\to\infty} (z^{-\mu}A, e^{s\rho}\mathcal{J}_{\tau-s\rho}(\alpha))_{\mathcal{H}^{\mathcal{X}}} \\ &= z^{p-n/2}(z^{-\mu}A, z^{n/2-p}\mathcal{J}^{\rm CY}_\tau(\alpha))_{H_0} \in z^{p-n/2}\mathbb{C}.\end{aligned}$$

Note that the last line is a period of $z^{n/2-p}\mathcal{J}^{\rm CY}_\tau(\alpha) \in F^{n-p}_\tau$. Therefore, *the A-model integral period (83) approaches to a period of the finite dimensional VHS in the conformal limit*. Note that in this limit, the integral period depends only on $\tau \in H^2_{\rm orb}(\mathcal{X})/\mathbb{C}\rho$.

Now we focus on integral periods for $F^n_\tau \subset H_0$. Note that $F^n_\tau = z^{n/2}\hat{\mathbb{F}}^{\rm CY} \cap H_0 = z^{n/2}\mathcal{J}^{\rm CY}_\tau(H^0_{\rm orb}(\mathcal{X}))$ is one dimensional over $\mathbb{C}$. We use the Galois action (monodromy action) to choose a good set of integral vectors in $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$. Let $L$ be a line bundle on $\mathcal{X}$ which is a pull-back of an ample line bundle on the coarse moduli space $X$ of $\mathcal{X}$. Then the Galois action $G^{\mathcal{V}}([L]) \in \operatorname{End}(\mathcal{V}^{\mathcal{X}})$ in (34) is unipotent since $f_v([L]) = 0$ for a pulled-back line bundle $L$. Take a weight filtration $W_k$ on $\mathcal{V}^{\mathcal{X}}$ defined by the logarithm $\operatorname{Log}(G^{\mathcal{V}}([L])) = -2\pi\sqrt{-1}c_1(L)$. See the proof of Proposition 3.6 for the weight filtration. This is given by (independent of a choice of $L$)

$$W_k = \bigoplus_{v\in\mathsf{T}} H^{\ge n_v - k}(\mathcal{X}_v). \tag{86}$$

The weight filtration is defined over $\mathbb{Q}$. We will also use the subspace $\operatorname{Ker}(H^2(\mathcal{X})) = \{\alpha \in \mathcal{V}^{\mathcal{X}} \ ; \ \tau_{0,2}\cdot\alpha = 0, \forall \tau_{0,2} \in H^2(\mathcal{X})\}$. Since this consists of $\alpha \in \mathcal{V}^{\mathcal{X}}$ satisfying $G^{\mathcal{V}}(\xi)\alpha = \alpha$ for every integral cohomology class $\xi \in H^2(X,\mathbb{Z})$ on the coarse moduli space, this is also defined over $\mathbb{Q}$. These subspaces define the following filtration on $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$:

$$(W_{-n}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}) \subset (\operatorname{Ker}(H^2(\mathcal{X}))\cap W_{-n+2}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}) \subset (W_{-n+2}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1})$$

which are full lattices of the vector spaces:

$$H^{2n}(\mathcal{X}) \subset H^{2n}(\mathcal{X}) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v) \subset (H^{\ge 2n-2}(\mathcal{X})\cap\operatorname{Ker}(\rho)) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v).$$

Note that by the Gorenstein assumption, there is no $v \in \mathsf{T}$ satisfying $n_v = n-1$ and that $n_v = n-2$ implies $\iota_v = 1$. Thus these subspaces are contained in $H^{\ge 2n-2}_{\rm orb}(\mathcal{X})$. We take integral vectors $A_0, A_1, \dots, A_\flat, A_{\flat+1}, \dots, A_\sharp$ in $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ compatible with this filtration:

$$\begin{aligned} W_{-n}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} &= \mathbb{Z}A_0, \\ \operatorname{Ker}(H^2(\mathcal{X}))\cap W_{-n+2}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} &= \mathbb{Z}A_0 + \textstyle\sum_{i=1}^{\flat}\mathbb{Z}A_i, \\ W_{-n+2}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} &= \mathbb{Z}A_0 + \textstyle\sum_{i=1}^{\flat}\mathbb{Z}A_i + \sum_{i=\flat+1}^{\sharp}\mathbb{Z}A_i. \end{aligned}$$

The vector $A_0$ is unique up to sign and invariant under all Galois action. In analogy with the Calabi-Yau B-model, we normalize a generator $\Omega_\tau \in F^n_\tau$ by the condition

$$(z^{-\mu}A_0, \Omega_\tau)_{H_0} = 1. \tag{87}$$

**Proposition 6.3.** *For $\tau \in H^2_{\rm orb}(\mathcal{X})$, we write $\tau = \tau_{0,2}+\tau_{\rm tw} = \tau_{0,2}+\tau'_{\rm tw}+\tau''_{\rm tw}$ with $\tau_{0,2} \in H^2(\mathcal{X})$, $\tau_{\rm tw} \in \bigoplus_{\iota_v=1} H^0(\mathcal{X}_v)$, $\tau'_{\rm tw} \in \bigoplus_{n_v=n-2} H^0(\mathcal{X}_v)$ and $\tau''_{\rm tw} \in \bigoplus_{n_v<n-2,\ \iota_v=1} H^0(\mathcal{X}_v)$. Define $a_i := (A_i, 1)_{\rm orb}$. Under the normalization (87), we have $\Omega_\tau = \sqrt{-1}^n a_0^{-1} z^{n/2}\mathcal{J}^{\rm CY}_\tau(1)$*

*and the integral periods* $(z^{-\mu}A_i, \Omega_\tau)_{H_0}$ *give an affine co-ordinate system on* $(H^2(\mathcal{X})/\mathbb{C}\rho)\oplus\bigoplus_{n_v=n-2} H^0(\mathcal{X}_v)$*:*

$$(z^{-\mu}A_i, \Omega_\tau)_{H_0} = a_0^{-1}a_i - (a_0^{-1}A_i, \tau'_{\rm tw})_{\rm orb}, \quad 1 \le i \le \flat,$$
$$(z^{-\mu}A_i, \Omega_\tau)_{H_0} = a_0^{-1}a_i - (a_0^{-1}A_i, \tau'_{\rm tw})_{\rm orb} - \frac{1}{2\pi\sqrt{-1}}[C_i]\cap\tau_{0,2}, \quad \flat+1 \le i \le \sharp$$

*where* $[C_i] \in H_2(\mathcal{X})$ *is the Poincaré dual of the* $H^{2n-2}(\mathcal{X})$*-component of* $2\pi\sqrt{-1}a_0^{-1}A_i$ *and*

$$[C_i] \in H_2(X,\mathbb{Z}) \cap \operatorname{Ker}\rho, \quad \textit{where } X \textit{ is the coarse moduli space of } \mathcal{X}. \tag{88}$$

$[C_{\flat+1}], \dots, [C_\sharp]$ *form a* $\mathbb{Q}$*-basis of* $H_2(X,\mathbb{Q})\cap\operatorname{Ker}\rho$*. The period for* $B \in \operatorname{Ker}(H^2(\mathcal{X}))\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ *is possibly non-linear and has the asymptotic*

$$(z^{-\mu}B, \Omega_\tau)_{H_0} \sim a_0^{-1}b - (a_0^{-1}B, \tau_{\rm tw})_{\rm orb}, \quad b := (B,1)_{\rm orb}$$

*as* $\tau$ *goes to the large radius limit point:*

$$\Re(\langle \tau_{0,2}, d\rangle) \to -\infty, \quad \forall d \in \operatorname{Eff}_{\mathcal{X}}\setminus\{0\}, \quad \tau_{\rm tw}\to 0.$$

*For the constant terms* $a_0^{-1}a_i$*,* $a_0^{-1}b$ *of integral periods, we have the following:*

(i) *If the following condition holds,*

$$\forall v \in \mathsf{T}\ (n_v = n-2 \implies \exists \xi \in H^2(\mathcal{X},\mathbb{Z}) \textit{ such that } f_v(\xi) > 0), \tag{89}$$

*we have* $a_0^{-1}a_i \in \mathbb{Q}$ *for* $1 \le i \le \flat$*.*

(ii) *If moreover* $H^*(\mathcal{X})$ *is generated by* $H^2(\mathcal{X})$ *as a ring and the following holds,*

$$\forall v \in \mathsf{T}\ (v \neq 0 \implies \exists \xi \in H^2(\mathcal{X},\mathbb{Z}) \textit{ such that } f_v(\xi) > 0), \tag{90}$$

*we have* $a_0^{-1}b \in \mathbb{Q}$ *for* $b = (B,1)_{\rm orb}$ *and* $B \in \operatorname{Ker}(H^2(\mathcal{X}))\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$*.*

(iii) *If the following holds for the integral structure,*

$$(H^{2n-2}_{\rm orb}(\mathcal{X}) \cap W_{-n+2} \cap \operatorname{Ker}\rho) \subset \mathcal{V}^{\mathcal{X}} \textit{ is defined over } \mathbb{Q}, \tag{91}$$

*we have* $a_0^{-1}a_i \in \mathbb{Q}$ *for* $1 \le i \le \sharp$*.*

*Proof.* By (85) and the string equation (see [2]), $\mathcal{J}^{\rm CY}_\tau(1)$ has the following expansions:

$$\mathcal{J}^{\rm CY}_\tau(1) = e^{\tau_{0,2}/z}\left(1 + \frac{\tau_{\rm tw}}{z} + \sum_{\substack{d\in\operatorname{Eff}_{\mathcal{X}}\cap\operatorname{Ker}(\rho),\\ l\ge 0,\\ d=0\Rightarrow l\ge 2.}} \sum_{i=1}^N \left\langle \tau_{\rm tw},\dots,\tau_{\rm tw}, \frac{\phi_i}{z(z-\psi)}\right\rangle^{\mathcal{X}}_{0,l+1d} e^{\langle\tau_{0,2},d\rangle}\phi^i\right)$$
$$= 1 + \frac{\tau}{z} + z^{-2}H^{\ge 4}_{\rm orb}(\mathcal{X})\otimes\mathbb{C}\{z^{-1}\}.$$

The forms of $\Omega_\tau$, $(z^{-\mu}A_i,\Omega_\tau)_{H_0}$ and $(z^{-\mu}B,\Omega_\tau)$ easily follow from these expansions. If $\xi \in H^2(X,\mathbb{Z})$ is an integral class on the coarse moduli space, we have $G^{\mathcal{V}}(\xi) = e^{-2\pi\sqrt{-1}\xi}$ by (34). Because the Galois action preserves the integral structure, $e^{-2\pi\sqrt{-1}\xi}A_i = A_i - m_iA_0$ for some integer $m_i$. Here, $2\pi\sqrt{-1}\xi A_i = m_iA_0$. Hence, $[C_i]\cap\xi = (2\pi\sqrt{-1}a_0^{-1}A_i,\xi)_{\rm orb} = a_0^{-1}(2\pi\sqrt{-1}\xi A_i, 1)_{\rm orb} = m_i \in \mathbb{Z}$. This shows (88). We set $V := H^{2n}(\mathcal{X}) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v) \subset \mathcal{V}^{\mathcal{X}}$. The Galois action preserves the full-lattice

$\mathbb{Z}A_0 + \sum_{i=1}^{\flat} \mathbb{Z}A_i$ of $V$. The Galois action on $V$ is simultaneously diagonalizable. Under the condition (89), $\mathbb{C}A_0$ gives the simultaneous eigenspace of eigenvalue 1 and $V' = \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v)$ gives the sum of simultaneous eigenspaces other than $\mathbb{C}A_0$. Then the direct sum decomposition

$$V = \mathbb{C}A_0 \oplus V'$$

is actually defined over $\mathbb{Q}$ since this is invariant under the Galois group over $\mathbb{Q}$. Therefore, there exists a rational number $c_i$ and $A_i' \in V'$ such that $A_i = c_i A_0 + A_i'$ for $1 \le i \le \flat$. Hence $a_i = (A_i, 1)_{\mathrm{orb}} = c_i(A_0, 1)_{\mathrm{orb}} = c_i a_0$. This shows (i). On the other hand, the Galois action on $\mathrm{Ker}(H^2(\mathcal{X}))$ is again simultaneously diagonalizable and preserves its full lattice $\mathrm{Ker}(H^2(\mathcal{X})) \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$. When $H^*(\mathcal{X})$ is generated by $H^2(\mathcal{X})$, we have $\mathrm{Ker}(H^2(\mathcal{X})) \cap H^*(\mathcal{X}) = H^{2n}(\mathcal{X})$ by Poincaré duality. Therefore, if moreover (90) holds, $H^{2n}(\mathcal{X}) = \mathbb{C}A_0$ is the simultaneous eigenspace of eigenvalue 1 of the Galois action on $\mathrm{Ker}(H^2(\mathcal{X}))$. For the same reason as above, the decomposition $\mathrm{Ker}(H^2(\mathcal{X})) = \mathbb{C}A_0 \oplus (\bigoplus_{v \in \mathsf{T}'} H^*(\mathcal{X}_v) \cap \mathrm{Ker}(H^2(\mathcal{X})))$ is defined over $\mathbb{Q}$. (ii) follows from this. The condition (91) implies the decomposition over $\mathbb{Q}$: $W_{-n+2} \cap \mathrm{Ker}\,\rho = H^{2n}(\mathcal{X}) \oplus (H^{2n-2}_{\mathrm{orb}}(\mathcal{X}) \cap W_{-n+2} \cap \mathrm{Ker}\,\rho)$. (iii) follows from this similarly. □

**Remark 6.4.** The conditions (89), (90) are weaker versions of (40). The condition (91) does not seem to follow from monodromy consideration. But this happens for the $\widehat{\Gamma}$-integral structures. See Example 6.5 below.

**Example 6.5.** Taking the $\widehat{\Gamma}$-integral structure in Definition-Proposition 3.16, we give explicit examples of A-model integral periods. By a natural map from the $K$-group of coherent sheaves to the $K$-group of topological orbifold vector bundles, we can regard a coherent sheaf as an element of $K(\mathcal{X})$. The integral vector $A_0 \in W_{-n} \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ is given by the image of the structure sheaf $\mathcal{O}_x$ of a non-stacky point $x \in \mathcal{X}$:

$$A_0 = \Psi([\mathcal{O}_x]) = \frac{(2\pi\sqrt{-1})^n}{(2\pi)^{n/2}}[\mathrm{pt}].$$

Here, we used the Poincaré duality to identify $[\mathrm{pt}] \in H_0(\mathcal{X})$ with an element in $H^{2n}(\mathcal{X})$. Hence we have $\Omega_\tau = (2\pi)^{-n/2} z^{n/2} \mathcal{J}^{\mathrm{CY}}_\tau(1)$.

(i) Let $\mathcal{X} = X$ be a manifold and $C \subset X$ be a smooth curve of genus $g$ such that $[C] \cap \rho = 0$. Then $[\mathcal{O}_C(g-1)]$ gives an integral vector $A_C \in W_{-n+2} \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$

$$A_C := \Psi([\mathcal{O}_C(g-1)]) = \frac{(2\pi\sqrt{-1})^{n-1}}{(2\pi)^{n/2}}[C]$$

and an integral period

$$(z^{-\mu} A_C, \Omega_\tau)_{H_0} = -\frac{1}{2\pi\sqrt{-1}}[C] \cap \tau.$$

(ii) Let $\Psi([V])$ be any integral vector in $W_{-n+2} \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$. Since $\widehat{\Gamma}_{\mathcal{X}}$ on the untwisted sector is of the form $1 - \gamma\rho +$higher degree ($\gamma$ is the Euler constant), it follows that the $H^{2n}(\mathcal{X})$-component of $\Psi([V])$ belongs to $(2\pi)^{-n/2}(2\pi\sqrt{-1})^n H^{2n}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}A_0$. This

implies that the component projection $W_{-n+2}\cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} \to H^{2n}(\mathcal{X}) = \mathbb{C}A_0$ is defined over $\mathbb{Q}$. Therefore, the condition (91) holds for the $\widehat{\Gamma}$-integral structure. We have

$$(z^{-\mu}\Psi([V]),\Omega_\tau)_{H_0} = \int_{\mathcal{X}} \operatorname{ch}(V) - (a_0^{-1}\Psi([V]),\tau'_{\rm tw})_{\rm orb} - \frac{1}{2\pi\sqrt{-1}}[C]\cap\tau_{0,2}.$$

for some $[C]\in H_2(X,\mathbb{Z})\cap \operatorname{Ker}\rho$ and $a_0 = (2\pi)^{-n/2}(2\pi\sqrt{-1})^n$.

(iii) Let $y\in\mathcal{X}$ be a possibly stacky point. Let $\clubsuit\colon \operatorname{Aut}(y)\to\operatorname{End}(V)$ be a finite dimensional representation of the automorphism group of $y$. This defines a coherent sheaf $\mathcal{O}_y\otimes V$ supported on $y$ and an integral vector $A_{(y,V)} := \Psi([\mathcal{O}_y\otimes V])\in \operatorname{Ker}(H^2(\mathcal{X}))\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$. Using Toen's Riemann-Roch formula [67], one calculates

$$A_{(y,V)} = \frac{(2\pi\sqrt{-1})^n}{(2\pi)^{n/2}}\sum_{(g)\subset\operatorname{Aut}(y)}\frac{(-1)^{n+n_{v(g)}+\iota_{v(g)}}\operatorname{Tr}(\clubsuit(g^{-1}))}{|C(g)|\prod_{j=1}^{n-n_{v(g)}}\Gamma(f_{g,j})}[\mathrm{pt}]_{v(g)},$$

where the sum is over all conjugacy classes $(g)$ of $g\in\operatorname{Aut}(y)$, $C(g)$ is the centralizer of $g$, $v(g)\in\mathsf{T}$ is the inertia component containing $(y,g)\in I\mathcal{X}$, $[\mathrm{pt}]_{v(g)}$ is the homology class of a point on $\mathcal{X}_{v(g)}$ (represented by a map $\mathrm{pt}\to\mathcal{X}_v$ of stacks), $f_{g,1},\dots f_{g,n-n_{v(g)}}$ are rational numbers in $(0,1)$ such that $\{e^{2\pi\sqrt{-1}f_{g,j}}\}_j$ is a multi-set of the eigenvalues $\neq 1$ of the $g$ action on $T_y\mathcal{X}$. The corresponding integral period has the asymptotic

$$(z^{-\mu}A_{(y,V)},\Omega_\tau)_{H_0}\sim\frac{\dim(V)}{|\operatorname{Aut}(y)|}+\sum_{\substack{(g)\subset\operatorname{Aut}(y)\\ \iota_{v(g)}=1}}\frac{\operatorname{Tr}(\clubsuit(g))}{|C(g)|\prod_{j=1}^{n-n_{v(g)}}\Gamma(1-f_{g,j})}[\mathrm{pt}]_{v(g)}\cap\tau_{\rm tw}$$

in the large radius limit. This asymptotic is exact if $y\notin\mathcal{X}_v$ for all $v$ with $\operatorname{codim}\mathcal{X}_v = n-n_v\geq 3$ or equivalently, $A_{(y,V)}\in\operatorname{Ker}(H^2(\mathcal{X}))\cap W_{-n+2}\cap\mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$.

6.3. **Ruan's conjecture with integral structure.** Yongbin Ruan's crepant resolution conjecture states that when $Y$ is a crepant resolution of the coarse moduli space $X$ of a Gorenstein orbifold $\mathcal{X}$,

$$\pi\colon Y\to X,\quad \pi^*(K_X) = K_Y,$$

the quantum cohomology for $Y$ and the orbifold quantum cohomology for $\mathcal{X}$ are related by analytic continuation in the quantum parameters. See [61, 13, 26] for references. In the joint work [25] with Coates and Tseng, in some examples of toric wall-crossings, we found the picture[8] that the A-model $\frac{\infty}{2}$VHS's of $Y$ and $\mathcal{X}$ are connected by analytic continuation and that the two $\frac{\infty}{2}$VHS's will match under a certain linear symplectic transformation $\mathbb{U}\colon\mathcal{H}^{\mathcal{X}}\to\mathcal{H}^Y$. This symplectic transformation $\mathbb{U}$ encodes all the information on relationships between the genus zero Gromov-Witten theories of $\mathcal{X}$ and $Y$. We refer the reader to [26] for a detailed discussion on the symplectic transformation and relationships to other versions of Ruan's conjecture. In this section, we incorporate integral structures into this picture and propose a possible relationship between the classical McKay correspondence and Ruan's conjecture.

Mirror symmetry and the integral structure calculation in Section 4 suggest the following refined picture involving $K$-groups:

[8] The symplectic transformation here also appeared in the work of Aganagic-Bouchard-Klemm [4] and was also conceived by Ruan himself.

(a) There exist "natural" integral structures on the (algebraic) A-model $\frac{\infty}{2}$VHS of $\mathcal{X}$ and $Y$. The corresponding integral lattices in $\mathcal{V}^{\mathcal{X}}$ and $\mathcal{V}^{Y}$ are given by the images of the $K$-groups of topological (resp. algebraic) orbifold vector bundles:

$$\Psi^{\mathcal{X}}\colon K(\mathcal{X}) \to \mathcal{V}^{\mathcal{X}}, \quad \Psi^{Y}\colon K(Y) \to \mathcal{V}^{Y}.$$

In the discussion below, we do not need to assume that $\Psi^{\mathcal{X}}$ and $\Psi^{Y}$ are defined by the same formula as $\Psi$ in the $\widehat{\Gamma}$-integral structure, but we assume that they satisfy the same conditions (i), (ii), (iii) in Definition-Proposition 3.16 as the $\widehat{\Gamma}$-integral structure satisfies.

(b) There exists an isomorphism of $K$-groups

$$\mathbb{U}_K\colon K(\mathcal{X}) \cong K(Y)$$

which preserves the Mukai pairing (as given in Definition-Proposition 3.16) and commutes with the tensor by a topological (resp. algebraic) line bundle $L$ on the coarse moduli space of $\mathcal{X}$, $\mathbb{U}_K(L\otimes\cdot) = \pi^*(L)\otimes\mathbb{U}_K(\cdot)$.

(c) Via $\Psi^{\mathcal{X}}$ and $\Psi^{Y}$, $\mathbb{U}_K$ induces an isomorphism $\mathbb{U}_{\mathcal{V}}\colon \mathcal{V}^{\mathcal{X}} \cong \mathcal{V}^{Y}$ preserving the pairing. By (32), $\mathcal{V}^{\mathcal{X}}$ (resp. $\mathcal{V}^{Y}$) is identified with the space of multi-valued flat sections of a flat bundle $(\mathsf{H}^{\mathcal{X}}, \widehat{\nabla}_{z\partial_z})$ (resp. $(\mathsf{H}^{Y}, \widehat{\nabla}_{z\partial_z})$) over $\mathbb{C}^*$. Because $\mathbb{U}_{\mathcal{V}}$ commutes with the monodromy transformation in $z$, $\mathbb{U}_{\mathcal{V}}$ induces a map of flat bundles $\mathbb{U}\colon (\mathsf{H}^{\mathcal{X}}, \widehat{\nabla}_{z\partial_z}) \to (\mathsf{H}^{Y}, \widehat{\nabla}_{z\partial_z})$. This is considered as a $\mathbb{C}\{z,z^{-1}\}$-linear symplectic isomorphism $\mathbb{U}\colon \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^{Y}$ (with respect to the symplectic form (30)) making the following diagram commute:

$$\begin{array}{ccc} K(\mathcal{X}) & \xrightarrow{\mathbb{U}_K} & K(Y) \\ {\scriptstyle z^{-\mu}z^{\rho}\Psi^{\mathcal{X}}}\big\downarrow & & \big\downarrow{\scriptstyle z^{-\mu}z^{\rho}\Psi^{Y}} \\ \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{\mathcal{X}}) & \xrightarrow{\mathbb{U}} & \Gamma(\widetilde{\mathbb{C}^*}, \mathsf{H}^{Y}). \end{array}$$

The isomorphism $\mathbb{U}$ so defined sends the A-model $\frac{\infty}{2}$VHS $\mathbb{F}^{\mathcal{X}}_{\tau} \subset \mathcal{H}^{\mathcal{X}}$ of $\mathcal{X}$ to that $\mathbb{F}^{Y}_{\tau} \subset \mathcal{H}^{Y}$ of $Y$, *i.e.*

$$\mathbb{U}(\mathbb{F}^{\mathcal{X}}_{\tau}) = \mathbb{F}^{Y}_{\Upsilon(\tau)}$$

where $\Upsilon$ is a map from a subdomain of $H^*_{\rm orb}(\mathcal{X})$ to a subdomain of $H^*(Y)$ where the quantum cohomology of $\mathcal{X}$ and $Y$ can be analytically continued respectively.

**Remark 6.6.** The isomorphism of $K$-groups (or even the equivalence of derived categories of coherent sheaves) are studied in the context of McKay correspondence and usually given by a Fourier-Mukai transformation. We expect that the isomorphism $\mathbb{U}_K$ in (b) will be given as a Fourier-Mukai transformation. In fact, Borisov-Horja [11] showed that an analytic continuation of solutions to the GKZ-system corresponds to a Fourier-Mukai transformation between $K$-groups of toric Calabi-Yau orbifolds. We can also ask if the integral structures have the same "functoriality" as the $K$-theory has. In this viewpoint, the map $\Psi^{\mathcal{X}}$ will play a role of "natural transformation" from $K$-theory integral structures to quantum cohomology.

We discuss what follows from this picture, assuming $\mathcal{X}$ is weak Fano, *i.e.* $c_1(\mathcal{X})$ is nef. Since this picture contains the suggestions we made in [25], it in particular implies that quantum cohomology of $\mathcal{X}$ and $Y$ are identified via $\Upsilon$ and $\mathbb{U}$ *as a family of algebras* (*not necessarily as Frobenius manifolds*). However, the large radius limit points for $\mathcal{X}$ and $Y$ are not identified under $\Upsilon$, so we need analytic continuations indeed. We will not repeat the argument in [25, 26] on the isomorphism of quantum cohomology algebras here. Let us first observe that integral periods of $\mathcal{X}$ and $Y$ in the conformal limit match under $\Upsilon$ and $\mathbb{U}$ (see (93) below). Because $\mathbb{U}_K$ commutes with the tensor by a line bundle pulled back from $X$, it follows that $\mathbb{U}$ must commute with $H^2(\mathcal{X})$ ((b), Conjecture 4.1 in [26]; (b), Section 5 in [25]), *i.e.*

$$\mathbb{U}(\alpha \cup \cdot) = \pi^*(\alpha) \cup \mathbb{U}(\cdot), \quad \alpha \in H^2(\mathcal{X}). \tag{92}$$

By definition, $\mathbb{U}$ commutes with $\widehat{\nabla}_{z\partial_z}$-action on $\mathcal{H}^{\mathcal{X}}$ and $\mathcal{H}^Y$. Hence by (92) and (29),

$$\mathbb{U} \circ (z\partial_z + \mu^{\mathcal{X}}) = (z\partial_z + \mu^Y) \circ \mathbb{U}$$

*i.e.* $\mathbb{U}$ is degree-preserving. Since $\mathcal{X}$ is weak Fano, by the discussion leading to Theorem 8.2 in [26] (essentially using Lemma 5.1 *ibid.*), we know that $\Upsilon$ should map $H^2_{\mathrm{orb}}(\mathcal{X})$ to $H^2(Y)$:

$$\Upsilon(H^2_{\mathrm{orb}}(\mathcal{X})) \subset H^2(Y).$$

The conformal limit $\tau \to \tau - s\rho$, $\Re(s) \to \infty$ on $H^2_{\mathrm{orb}}(\mathcal{X})$ should also be mapped to the conformal limit on $H^2(Y)$ under $\Upsilon$ because this flow is generated by the Euler vector field and the two Euler vector fields should match under $\Upsilon$ (the Euler vector field is a part of the data of $\frac{\infty}{2}$VHS). Therefore, by (92) and $\pi^* c_1(\mathcal{X}) = c_1(Y)$, the $\frac{\infty}{2}$VHSs appearing in the conformal limit (see Definition 6.1) also match under $\mathbb{U}$:

$$\mathbb{U}(\mathbb{F}^{\mathcal{X},\mathrm{CY}}_\tau) = \mathbb{F}^{Y,\mathrm{CY}}_{\Upsilon(\tau)}.$$

In particular, the finite dimensional VHS's $(F^{\mathcal{X},\bullet}_\tau \subset H^{\mathcal{X}}_0)$, $(F^{Y,\bullet}_\tau \subset H^Y_0)$ associated with these also match:

$$\mathbb{U}(F^{\mathcal{X},\bullet}_\tau) = F^{Y,\bullet}_{\Upsilon(\tau)}, \quad \mathbb{U} \colon \mathcal{H}^{\mathcal{X}} \supset H^{\mathcal{X}}_0 \to H^Y_0 \subset \mathcal{H}^Y.$$

We used that $\mathbb{U}$ induces a map from $H^{\mathcal{X}}_0 = \mathrm{Ker}(z\partial_z + \mu^{\mathcal{X}})$ to $H^Y_0 = \mathrm{Ker}(z\partial_z + \mu^Y)$. Let $L$ be an ample line bundle on $X$. Consider the weight filtration $W^{\mathcal{X}}_k$ (86) on $\mathcal{V}^{\mathcal{X}}$ defined by the Galois action logarithm $-2\pi\sqrt{-1}c_1(L)$. The first term $W^{\mathcal{X}}_{-n}$ of the weight filtration is given by $\mathrm{Im}(c_1(L)^n)$. Thus $\mathbb{U}_{\mathcal{V}}(W^{\mathcal{X}}_{-n}) = \mathrm{Im}(\pi^*(c_1(L))^n) = H^{2n}(Y)$. Note that $\pi^*(c_1(L))^n$ is non-trivial since $\pi \colon Y \to X$ is birational. Therefore, for the weight filtration $W^Y_k$ on $\mathcal{V}^Y$ (defined similarly by the Galois action logarithm corresponding to an ample line bundle on $Y$), we have

$$\mathbb{U}_{\mathcal{V}}(W^{\mathcal{X}}_{-n}) = W^Y_{-n}.$$

As we did before, we use an integral vector $A^{\mathcal{X}}_0$ (unique up to sign) in $W^{\mathcal{X}}_{-n} \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$ to normalize a generator $\Omega^{\mathcal{X}}_\tau \in F^{\mathcal{X},n}_\tau$ and then use $A^Y_0 := \mathbb{U}_{\mathcal{V}}(A^{\mathcal{X}}_0) \in W^Y_{-n} \cap \mathcal{V}^Y_{\mathbb{Z},1}$ to normalize $\Omega^Y_\tau \in F^{Y,n}_\tau$ (see (87)). Because the $\mathbb{U}$ preserves the pairing, we have

$$\mathbb{U}(\Omega^{\mathcal{X}}_\tau) = \Omega^Y_{\Upsilon(\tau)}.$$

When $A^{\mathcal{X}} \in \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1} = \mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} \cap \operatorname{Ker}(c_1(\mathcal{X}))$, the corresponding vector $A^Y = \mathbb{U}_{\mathcal{V}}(A^{\mathcal{X}})$ belongs to $\mathcal{V}^Y_{\mathbb{Z}} \cap \operatorname{Ker}(\pi^*(c_1(\mathcal{X}))) = \mathcal{V}^Y_{\mathbb{Z},1}$ and the integral periods match

$$(z^{-\mu}A^{\mathcal{X}}, \Omega^{\mathcal{X}}_{\tau})_{H_0^{\mathcal{X}}} = (z^{-\mu}A^Y, \Omega^Y_{\Upsilon(\tau)})_{H_0^Y}. \tag{93}$$

Now we can make predictions on the specialization values of quantum parameters. $\operatorname{Ker}(\pi^*H^2(\mathcal{X})) \subset \mathcal{V}^Y$ is defined over $\mathbb{Q}$ since this is the intersection of $\operatorname{Ker}(\operatorname{id} - G^{\mathcal{V}}(\pi^*\xi))$ over integral class $\xi \in H^2(X,\mathbb{Z})$. Take a basis $A_0^Y, A_1^Y, \dots, A_\natural^Y$ of $\operatorname{Ker}(\pi^*H^2(\mathcal{X})) \cap W^Y_{-n+2} \cap \mathcal{V}^Y_{\mathbb{Z},1}$. These generate a full lattice in $H^{2n}(Y) \oplus (H^{2n-2}(Y) \cap \operatorname{Ker}\pi_*)$ over $\mathbb{C}$. By Proposition 6.3, the integral periods for $A_1^Y, \dots, A_\natural^Y$ are of the form:

$$(z^{-\mu}A_i^Y, \Omega^Y_{\tau})_{H_0^Y} = a_0^{-1}a_i - \frac{1}{2\pi\sqrt{-1}}[C_i] \cap \tau, \quad a_i := (A_i^Y, 1), \tag{94}$$

for some $[C_i] \in H_2(Y,\mathbb{Z}) \cap \operatorname{Ker}\pi_*$. $[C_1], \dots, [C_\natural]$ are a $\mathbb{Q}$-basis of $H_2(Y,\mathbb{Q}) \cap \operatorname{Ker}\pi_*$, so these form an affine co-ordinate system on $H^2(Y)/\operatorname{Im}\pi^*$. The integral vector $A_i^{\mathcal{X}}$ corresponding to $A_i^Y$ belongs to $\operatorname{Ker}(H^2(\mathcal{X})) \cap \mathcal{V}^{\mathcal{X}}_{\mathbb{Z},1}$. From (93), Proposition 6.3 and Example 6.5, our picture leads to the following prediction:

(i) Assume that $H^*(\mathcal{X})$ is generated by $H^2(\mathcal{X})$ and that the condition (90) is satisfied. Then the integral periods of $Y$ of the form (94) take rational values at the large radius limit point of $\mathcal{X}$.

(ii) Assume in addition to (i) that the condition (91) holds for $Y$. Then $a_0^{-1}a_i$ above is rational, so the "quantum parameter" $q_C := \exp([C] \cap \tau)$ with $[C] \in H_2(Y,\mathbb{Z}) \cap \operatorname{Ker}\pi_*$ for $Y$ specializes to a root of unity at the large radius limit point of $\mathcal{X}$.

(iii) Let $C \subset Y$ be a smooth rational curve in the exceptional set. Assume in addition to (iii) that $\mathbb{U}_K^{-1}$ sends $[\mathcal{O}_C(-1)] \in K(Y)$ to $[\mathcal{O}_x \otimes V] \in K(\mathcal{X})$ for $x = \pi(C)$ and some representation $V$ of $\operatorname{Aut}(x)$. Then the quantum parameter $q_C$ specializes to $\exp(-2\pi\sqrt{-1}\dim V/|\operatorname{Aut}(x)|)$ at the large radius limit point of $\mathcal{X}$.

For the $A_n$ singularity resolution, each irreducible curve in the exceptional set corresponds to a one-dimensional irreducible representation of $\mathbb{Z}/(n+1)\mathbb{Z}$ under McKay correspondence. If we use this McKay correspondence as $\mathbb{U}_K$, the prediction of specialization values made in (iii) is true [21]. Also, under the McKay correspondence, (iii) gives the same prediction (up to complex conjugation) made by Bryan-Graber [13] and Bryan-Gholampour [12] for the ADE surface singularities and $\mathbb{C}^3/G$ with a finite subgroup $G \subset SO(3)$.

## 7. Appendix

7.1. **Proof of (52).** Birkhoff's theorem implies that there exists an open dense neighborhood of $\mathbf{1}$ in the loop group $LGL_N(\mathbb{C})$ which is diffeomorphic to the product of subgroups $L_1^+GL_N(\mathbb{C}) \times L^-GL_N(\mathbb{C})$ [60]. We use the inverse function theorem for Hilbert manifolds to explain the order estimate in (52). Consider the space $LGL_N(\mathbb{C})^{1,2}$ of Sobolev loops which consists of maps $\lambda\colon S^1 \to GL_N(\mathbb{C})$ such that $\lambda$ and its weak derivative $\lambda'$ are square integrable. Note that this is a subgroup of the group of

continuous loops by Sobolev embedding theorem $W^{1,2}(S^1) \subset C^0(S^1)$ and the multiplication theorem $W^{1,2}(S^1) \times W^{1,2}(S^1) \to W^{1,2}(S^1)$. $LGL_N(\mathbb{C})^{1,2}$ is a Hilbert manifold modeled on the Hilbert space $W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$. A co-ordinate chart of a neighborhood of $\mathbf{1}$ is given by the exponential map $A(z) \mapsto e^{A(z)}$. Let $L_1^+ GL_N(\mathbb{C})^{1,2}$ be the subgroup of $LGL_N(\mathbb{C})^{1,2}$ consisting of the boundary values of holomorphic maps $\lambda_+ \colon \{|z| < 1\} \to GL_N(\mathbb{C})$ satisfying $\lambda_+(0) = \mathbf{1}$. Let $L^- GL_N(\mathbb{C})^{1,2}$ be the subgroup of $LGL_N(\mathbb{C})^{1,2}$ consisting of the boundary values of holomorphic maps $\lambda_- \colon \{|z| > 1\} \cup \{\infty\} \to GL_N(\mathbb{C})$. Notice that $W^{1,2} := W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$ has the direct sum decomposition:

$$W^{1,2} = W^{1,2}_+ \oplus W^{1,2}_-, \tag{95}$$

where $W^{1,2}_+$ ($W^{1,2}_-$) is the closed subspace of $W^{1,2}(S^1, \mathfrak{gl}_N(\mathbb{C}))$ consisting of strictly positive Fourier series $\sum_{n>0} a_n z^n$ (non-positive Fourier series $\sum_{n \le 0} a_n z^n$ resp.) with $a_n \in \mathfrak{gl}_N(\mathbb{C})$. The subgroups $L_1^+ GL_N(\mathbb{C})^{1,2}$ and $L^- GL_N(\mathbb{C})^{1,2}$ are modeled on the Hilbert spaces $W^{1,2}_+$ and $W^{1,2}_-$ respectively. Consider the multiplication map $L_1^+ GL_N(\mathbb{C})^{1,2} \times L^- GL_N(\mathbb{C})^{1,2} \to LGL_N(\mathbb{C})^{1,2}$. The differential of this map at the identity is given by the sum $W^{1,2}_+ \times W^{1,2}_- \to W^{1,2}$ and is clearly an isomorphism. By the inverse function theorem for Hilbert manifolds, there exists a differentiable inverse map on a neighborhood of $\mathbf{1}$. In the case at hand, we have $\|(B_t^{-1} Q_t B_t)(C_t \overline{Q}_t C_t^{-1}) - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$ as $t \to \infty$. Therefore, this admits the Birkhoff factorization (52) for $t \gg 0$ with $\|\tilde{B}_t - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$ and $\|\tilde{C}_t - \mathbf{1}\|_{W^{1,2}} = O(e^{-\epsilon t})$. By Sobolev embedding, the order estimates hold also for the $C^0$-norm. (The method here does not work directly for the Banach manifold of continuous loops, since the decomposition (95) is not true in this case.)

7.2. **Proof of Lemma 4.8.** Let $B \subset \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ be a compact set. We need to show that $B' = \{(q, z, y) \ ; \ (q, z) \in B, \ y \in Y_q, \ \|df_{q,z}(y)\| \le \epsilon\}$ is compact. Assume that there exists a divergent sequence $\{(q_{(k)}, z_{(k)}, y_{(k)})\}_{k=0}^{\infty}$ in $B'$, *i.e.* any subsequence of it does not converge. Take an arbitrary Hermitian norm $\| \cdot \|$ on $\boldsymbol{N} \otimes \mathbb{C}$. Note that we have

$$\|df_{q,z}(y)\| = \frac{1}{|z|} \|\sum_{i=1}^{m} q^{\ell_i} y^{b_i} b_i\|.$$

By passing to a subsequence and renumbering $b_1, \dots, b_m$, we can assume that $q_{(k)}$ and $z_{(k)}$ converge and that $|y_{(k)}^{b_1}| \ge |y_{(k)}^{b_2}| \ge \cdots \ge |y_{(k)}^{b_m}|$ for all $k$. Since 0 is in the interior of $\hat{S}$, there exist $c_i > 0$ such that $\sum_{i=1}^m c_i b_i = 0$. Hence $\prod_{i=1}^m |y_{(k)}^{b_i}|^{c_i} = 1$. Because $y_{(k)}$ diverges, we must have $\lim_{k \to \infty} |y_{(k)}^{b_1}| = \infty$. Since $\|df_{q_{(k)}, z_{(k)}}(y_{(k)})\|$ is bounded, we have

$$0 = \lim_{k \to \infty} \frac{|z_{(k)}|}{|y_{(k)}^{b_1}|} \|df_{q_{(k)}, z_{(k)}}(y_{(k)})\| = \lim_{k \to \infty} \|\sum_{i=1}^{m} q_{(k)}^{\ell_i} y_{(k)}^{b_i - b_1} b_i\|.$$

Because $|y_{(k)}^{b_i - b_1}| \le 1$, by passing to a subsequence again, we can assume that $y_{(k)}^{b_i - b_1}$ converges to $\alpha_i \neq 0$ for all $1 \le i \le l$ and $y_{(k)}^{b_i - b_1}$ goes to 0 for $i > l$. Then we have

$$0 = \sum_{i=1}^{l} \tilde{q}^{\ell_i} \alpha_i b_i, \quad \tilde{q} = \lim_{k \to \infty} q_{(k)} \in \mathcal{M}^{\mathrm{o}}.$$

Put $\xi_{(k),i} := \log y_{(k),i}$. By choosing a suitable branch of the logarithm, we can assume that $\lim_{k\to\infty}\langle \xi_{(k)}, b_i - b_1\rangle = \log \alpha_i$ for $1 \le i \le l$ and $\lim_{k\to\infty}\langle \Re(\xi_{(k)}), b_i - b_1\rangle = -\infty$ for $i > l$. Let $V$ be the $\mathbb{C}$ subspace of $\boldsymbol{N}\otimes\mathbb{C}$ spanned by $b_i - b_1$ with $1 \le i \le l$. Take the orthogonal decomposition $\boldsymbol{N}\otimes\mathbb{C} \cong V \oplus V^\perp$ and write $\xi_{(k)} = \xi'_{(k)} + \xi''_{(k)}$, where $\xi'_{(k)} \in V$ and $\xi''_{(k)} \in V^\perp$. Then $\xi'_{(k)}$ converges to some $\xi' \in V$. Putting $\tilde{y}_i = \exp(\xi'_i)$, we have $\tilde{y}^{b_i - b_1} = \alpha_i$ for $1 \le i \le l$ and so

$$\sum_{i=1}^{l} \tilde{q}^{\ell_i}\tilde{y}^{b_i} b_i = \tilde{y}^{b_1}\Big(\sum_{i=1}^{l}\tilde{q}^{\ell_i}\tilde{y}^{b_i-b_1}b_i\Big) = 0. \tag{96}$$

On the other hand, for a sufficiently big $k$, $\langle \Re(\xi''_{(k)}), b_i - b_1\rangle = 0$ for $1 \le i \le l$ and $\langle \Re(\xi''_{(k)}), b_i - b_1\rangle < 0$ for $i > l$. This means that $b_1, \dots, b_l$ are on some face $\Delta$ of $\hat{S}$. But the equation (96) shows that $\tilde{y}$ is a critical point of $W_{\tilde{q},\Delta}$. This contradicts to the assumption that $W_{\tilde{q}}$ is non-degenerate at infinity.

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